

Nonlocal elliptic problems with nonlinear argument transformations near the points of conjugation

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Abstract

We consider elliptic equations of order $2m$ in a domain $G \subset \mathbb{R}^n$ with nonlocal conditions that connect the values of the unknown function and its derivatives on $(n-1)$ -dimensional submanifolds Υ_i (where $\bigcup_i \Upsilon_i = \partial G$) with the values on $\omega_{is}(\overline{\Upsilon_i}) \subset \overline{G}$. Nonlocal elliptic problems in dihedral angles arise as model problems near the conjugation points $g \in \overline{\Upsilon_i} \cap \Upsilon_j \neq \emptyset$, $i \neq j$. We study the case where the transformations ω_{is} correspond to nonlinear transformations in the model problems. It is proved that the operator of the problem remains Fredholm and its index does not change as we pass from linear argument transformations to nonlinear ones.

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Introduction

The first mathematicians who studied ordinary differential equations with nonlocal conditions were Sommerfeld [1], Tamarkin [2], Picone [3]. In 1932, Carleman [4] considered the problem of finding a holomorphic function in a bounded domain G , satisfying the following condition: the value of the unknown function at each point x of the boundary is connected with the value at $\omega(x)$, where $\omega(\omega(x)) = x$, $\omega(\partial G) = \partial G$. Such a statement of the problem originated further investigations of

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nonlocal elliptic problems with the shifts mapping the boundary onto itself. In 1969, Bitsadze and Samarskii [5] considered essentially different type of nonlocal problems. They studied the Laplace equation in a bounded domain G with the boundary-value condition connecting the values of the unknown function on a manifold $\Upsilon_1 \subset \partial G$ with the values on some manifold inside G ; on the set $\partial G \setminus \Upsilon_1$ the Dirichlet condition was imposed. In a general case, such a problem was formulated as an unsolved one.

The most difficult situation in the theory of nonlocal problems is that where the support of nonlocal terms intersects with the boundary of domain. We consider the following example. Let $G \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with the boundary $\partial G = \Upsilon_1 \cup \Upsilon_2 \cup \mathcal{K}_1$, where Υ_i are smooth open (in the topology of ∂G) $(n-1)$ -dimensional C^∞ -manifolds, $\mathcal{K}_1 = \bar{\Upsilon}_1 \cap \bar{\Upsilon}_2$ is an $(n-2)$ -dimensional connected C^∞ -manifold without a boundary. (If $n = 2$, then $\mathcal{K}_1 = \{g_1, g_2\}$, where g_1, g_2 are the ends of the curves $\bar{\Upsilon}_1, \bar{\Upsilon}_2$.) Suppose that, in a neighborhood of each point $g \in \mathcal{K}_1$, the domain G is diffeomorphic to some n -dimensional dihedral angle (plain angle if $n = 2$). In the domain G , we consider the nonlocal problem

$$\Delta u = f_0(y) \quad (y \in G), \quad (0.1)$$

$$u|_{\Upsilon_i} - b_i u(\omega_i(y))|_{\Upsilon_i} = 0 \quad (i = 1, 2). \quad (0.2)$$

Here $b_1, b_2 \in \mathbb{R}$; ω_i is an infinitely differentiable transformation mapping some neighborhood \mathcal{O}_i of the manifold Υ_i onto the set $\omega(\mathcal{O}_i)$ so that $\omega_i(\Upsilon_i) \subset G$, $\overline{\omega_i(\Upsilon_i)} \cap \partial G \neq \emptyset$, see figures 0.1.a and 0.1.b.

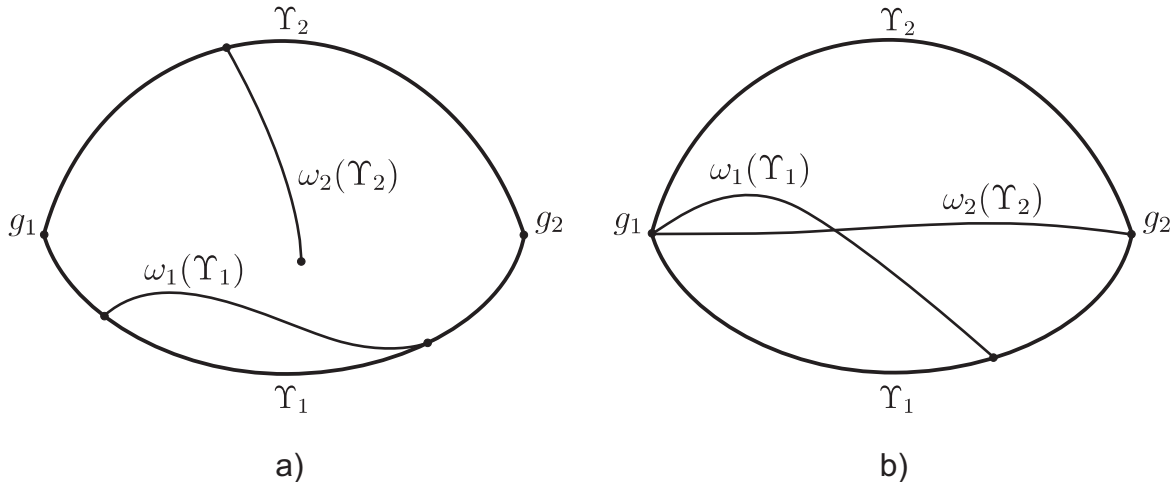


Figure 0.1: The domain G with the boundary $\partial G = \bar{\Upsilon}_1 \cup \bar{\Upsilon}_2$ for $n = 2$.

Problems of type (0.1), (0.2) were considered by many mathematicians (see [6, 7, 8] and others). The most complete theory for such problems is developed by Skubachevskii and his pupils [9, 10, 11, 12, 13, 14]. In particular, Fredholm solvability of higher-order elliptic equations with general nonlocal conditions is proved, asymptotics for solutions near the points of conjugation of nonlocal conditions is established, smoothness of solutions is studied. It is shown [15] that the index of nonlocal problem is equal to the index of the corresponding local one if the support of nonlocal terms does not intersect with the points of conjugation (see Fig. 0.1.a with g_1 and g_2 being the points of conjugation). Otherwise (see Fig. 0.1.b), this is not true.

Properties of nonlocal problems in bounded domains are essentially determined by properties of model nonlocal problems in dihedral (plain if $n = 2$) angles $\Omega = \{x = (y, z) \in \mathbb{R}^n : b' < \varphi <$

$b'', z \in \mathbb{R}^{n-2}$ corresponding to the points of conjugation of nonlocal conditions $((\varphi, r)$ are polar coordinates of y). Until now [9, 10, 11], it was studied the case where the transformations ω_{is} corresponded to linear transformations (i.e., compositions of rotation and expansion in y -plane) in model problems. However, such a restriction is quite unnatural in applications. Let us explain this on examples. Problem of type (0.1), (0.2) is a mathematical model for some plasma process in a bounded domain [16]. Nonlocal conditions connect the plasma temperature on the boundary of the domain with the temperature inside the domain and at other points of the boundary.

Another important application arises in the theory of diffusion processes. Such processes describe, for example, the Brownian motion of a particle in the membrane $G \subset \mathbb{R}^n$. It is known [17, 18, 19] that every diffusion process generates some Feller semigroup. By virtue of the Hille–Iosida theorem, the investigation of this semigroup may be reduced to the study of an elliptic operator with boundary-value conditions containing an integral over \bar{G} with respect to a non-negative Borel measure [20]. In the most difficult case where the measure is atomic, nonlocal conditions assume the form (0.2). Their probabilistic sense is as follows: once the particle gets to a point $y \in \Upsilon_i$, it either jumps to the point $\omega_i(y)$ with probability b_i ($0 \leq b_i \leq 1$) or “dies” with probability $1 - b_i$ (in this case, the process terminates). In general, both in the plasma theory and theory of diffusion processes, nonlinear argument transformations appear.

Let us mention one more application of nonlocal problems. In the monograph [21], it is shown that in some cases a boundary-value problem for elliptic differential-difference equation (in particular, arising in modern aircraft technology and modelling sandwich shells and plates [22, 21]) can be reduced to an elliptic equation with nonlocal conditions on shifts of the boundary. Thus, we again obtain nonlinear transformations. (These transformations are linear only if the boundary of domain coincides, on certain sets, with $(n - 1)$ -dimensional hyperplanes.)

Other applications and references to papers devoted to nonlocal problems can be found in [21].

In this paper, we consider an elliptic $2m$ -order equation in a domain $G \subset \mathbb{R}^n$ with nonlocal conditions connecting the values of the unknown function and its derivatives on $(n - 1)$ -dimensional manifolds Υ_i (where $\bigcup_i \Upsilon_i = \partial G$) with the values on $\omega_{is}(\Upsilon_i) \subset G$. As we mentioned before, the essential difficulties arise in the case where the support of nonlocal terms $\bigcup_{i,s} \overline{\omega_{is}(\Upsilon_i)}$ intersects with the boundary of domain. In this situation, the generalized solutions may have power singularities near some set [9]. (For example, in case of problem (0.1), (0.2), these singularities may appear near the points g_1 and g_2 .) Therefore, it is natural to consider such problems in weighted spaces. This allows one to investigate higher-order elliptic equations with general nonlocal conditions. We study the case where the transformations ω_{is} correspond to nonlinear transformations in model problems. It turns out that the problem with nonlinear transformation is neither a small nor compact perturbation of the corresponding local problem. Nevertheless, we show that, when passing from linear transformations to nonlinear ones, the operator of the problem remains Fredholm and its index does not change.

Notice that a more general structure of the conjugation points and nonlocal terms for second-order elliptic equations with nonlocal perturbations of the Dirichlet problem was considered in [8]. This also justifies the importance of nonlinear transformations ω_{is} . From our point of view, the advantage of the approach suggested is that it allows us to study $2m$ -order elliptic equations with general boundary-value conditions, nonlocal perturbations of which may be arbitrary large. On the other hand, this approach also allows us to investigate the asymptotic behavior of solutions near the conjugation points [9, 14].

Our paper is organized as follows. In § 1, we consider the statement of the problem and

discuss the conditions imposed on the argument transformations in nonlocal terms. Ibidem, we introduce basic functional spaces (Sobolev spaces with a weight) and obtain model problems in dihedral and plain angles. In § 2, we give an example of nonlocal problem with nonlinear argument transformation and show that the operator corresponding to this problem is neither a small nor compact perturbation of the operator corresponding to the problem with linearized transformations. In § 3, we study some properties of nonlinear transformations near the points of conjugation of nonlocal conditions and prove a number of lemmas which are used in § 4 for getting a priori estimates of solutions. In § 5, we construct a right regularizer, which, being combined with the a priori estimate, guarantees the Fredholm solvability of the nonlocal problem. Finally, in § 6, we show that the index of the problem with nonlinear argument transformations is equal to that of the problem with the transformations linearized near the points of conjugation of nonlocal conditions.

1 Statement of the problem in a bounded domain

1. Let $G \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with the boundary $\partial G = \bigcup_{i=1}^{N_0} \bar{\Upsilon}_i$, where Υ_i are smooth open (in the topology of ∂G) $(n-1)$ -dimensional C^∞ -manifolds. We assume that, in a neighborhood of each point $g \in \partial G \setminus \bigcup_{i=1}^{N_0} \Upsilon_i$, the domain G is diffeomorphic to some n -dimensional dihedral (plain if $n = 2$) angle $\Omega = \{x = (y, z) \in \mathbb{R}^n : 0 < b' < \varphi < b'' < 2\pi, z \in \mathbb{R}^{n-2}\}$, where (φ, r) are polar coordinates of y .

We denote by $\mathbf{P}(x, D)$ and $B_{i\mu s}(x, D)$ differential operators of order $2m$ and $m_{i\mu}$ respectively with complex-valued C^∞ -coefficients ($i = 1, \dots, N_0$; $\mu = 1, \dots, m$; $s = 0, \dots, S_i$). Let the operators $\mathbf{P}(x, D)$ and $B_{i\mu 0}(x, D)$ satisfy the following conditions (see, for example, [23, Chapter 2, § 1]).

Condition 1.1. *For all $x \in \bar{G}$, the operator $\mathbf{P}(x, D)$ is properly elliptic.*

Condition 1.2. *For all $i = 1, \dots, N_0$ and $x \in \bar{\Upsilon}_i$, the system $\{B_{i\mu 0}(x, D)\}_{\mu=1}^m$ covers the operator $\mathbf{P}(x, D)$.*

Let ω_{is} ($i = 1, \dots, N_0$; $s = 1, \dots, S_i$) be an infinitely differentiable transformation mapping some neighborhood \mathcal{O}_i of the manifold Υ_i onto the manifold $\omega_{is}(\mathcal{O}_i)$ so that $\omega_{is}(\Upsilon_i) \subset G$. We assume that the set

$$\mathcal{K} = \left\{ \bigcup_i (\bar{\Upsilon}_i \setminus \Upsilon_i) \right\} \cup \left\{ \bigcup_{i,s} \omega_{is}(\bar{\Upsilon}_i \setminus \Upsilon_i) \right\} \cup \left\{ \bigcup_{j,p} \bigcup_{i,s} \omega_{jp}(\omega_{is}(\bar{\Upsilon}_i \setminus \Upsilon_i) \cap \Upsilon_j) \right\}$$

can be represented in the form $\mathcal{K} = \bigcup_{j=1}^3 \mathcal{K}_j$, where

$$\mathcal{K}_1 = \bigcup_{p=1}^{N_1} \mathcal{K}_{1p} = \partial G \setminus \bigcup_{i=1}^{N_0} \Upsilon_i, \quad \mathcal{K}_2 = \bigcup_{p=1}^{N_2} \mathcal{K}_{2p} \subset \bigcup_{i=1}^{N_0} \Upsilon_i, \quad \mathcal{K}_3 = \bigcup_{p=1}^{N_3} \mathcal{K}_{3p} \subset G. \quad (1.1)$$

Here \mathcal{K}_{jp} are disjoint $(n-2)$ -dimensional connected C^∞ -manifolds without a boundary (points if $n = 2$).

We consider the nonlocal boundary-value problem

$$\mathbf{P}(x, D)u = f_0(x) \quad (x \in G), \quad (1.2)$$

$$\begin{aligned} \mathbf{B}_{i\mu}(x, D)u &\equiv \sum_{s=0}^{S_i} (B_{i\mu s}(x, D)u)(\omega_{is}(x))|_{\Upsilon_i} = g_{i\mu}(x) \\ (x \in \Upsilon_i; i = 1, \dots, N_0; \mu = 1, \dots, m), \end{aligned} \quad (1.3)$$

where $(B_{i\mu s}(x, D)u)(\omega_{is}(x)) = B_{i\mu s}(x', D_{x'})u(x')|_{x'=\omega_{is}(x)}$, $\omega_{i0}(x) \equiv x$.

Example 1.1. Let us consider problem (0.1), (0.2) in two-dimensional case, with the transformations ω_i corresponding to Fig. 1.1. Then we have $\mathcal{K}_1 = \{g_1, g_2\}$, $\mathcal{K}_2 = \{\omega_1(g_2)\}$,

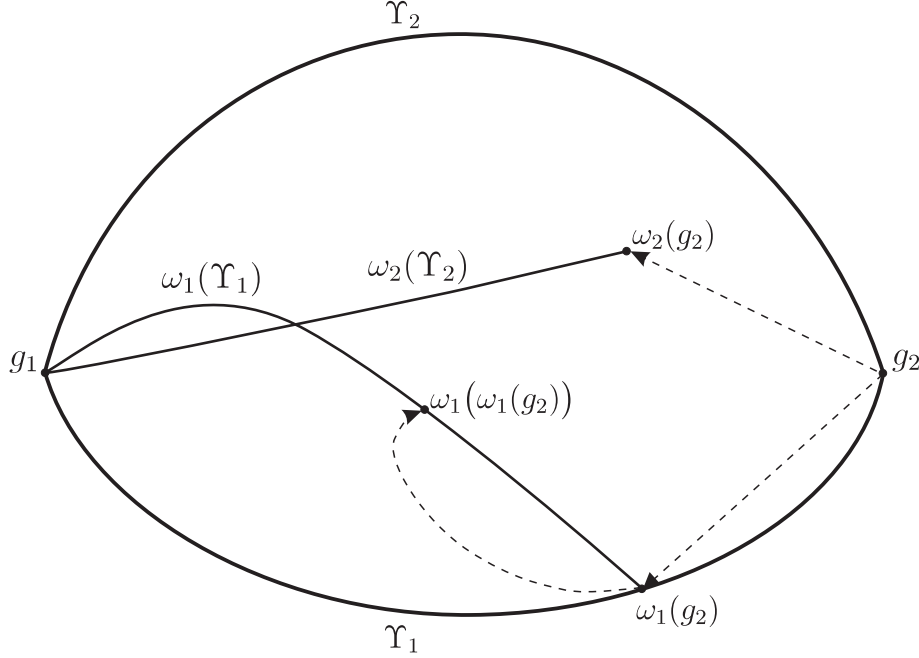


Figure 1.1: The domain G with the boundary $\partial G = \Upsilon_1 \cup \Upsilon_2$, $n = 2$.

$$\mathcal{K}_3 = \{\omega_2(g_2), \omega_1(\omega_1(g_2))\}.$$

In [9], it is shown that the solutions for problem (1.2), (1.3) may have power singularities near the points of the set \mathcal{K}_1 . Therefore, it is natural to consider problem (1.2), (1.3) in weighted spaces. We introduce the space $H_b^l(Q)$ as a completions of the set $C_0^\infty(\bar{Q} \setminus M)$ with respect to the norm

$$\|u\|_{H_b^l(Q)} = \left(\sum_{|\alpha| \leq l} \int_Q \rho^{2(b-|\alpha|)} |D^\alpha u|^2 dx \right)^{1/2}.$$

Here Q is the domain G , angle Ω , or \mathbb{R}^n ; $M = \mathcal{K}_1$ if $Q = G$ and $M = \{x = (y, z) \in \mathbb{R}^n : y = 0, z \in \mathbb{R}^{n-2}\}$ if $Q = \Omega$ or $Q = \mathbb{R}^n$; $C_0^\infty(\bar{Q} \setminus M)$ is the set of infinitely differentiable functions with compact supports being subsets of $\bar{Q} \setminus M$; $l \geq 0$ is an integer; $b \in \mathbb{R}$; $\rho = \rho(x) \in C^\infty(\mathbb{R}^n \setminus \mathcal{K}_1)$ is a function¹ satisfying $c_1 \text{dist}(x, \mathcal{K}_1) \leq \rho(x) \leq c_2 \text{dist}(x, \mathcal{K}_1)$ ($x \in G$, $c_1, c_2 > 0$, $\text{dist}(x, \mathcal{K}_1)$ denotes

¹The existence of the function $\rho(x)$ follows from Theorem 2 [24, Chapter 6, § 2].

the distance from x to \mathcal{K}_1) if $Q = G$ and $\rho(x) = |y|$ if $Q = \Omega$ or $Q = \mathbb{R}^n$. For $l \geq 1$, we denote by $H_b^{l-1/2}(\Upsilon)$ the space of traces on a smooth $(n-1)$ -dimensional manifold $\Upsilon \subset \bar{Q}$ with the norm

$$\|\psi\|_{H_b^{l-1/2}(\Upsilon)} = \inf \|u\|_{H_b^l(Q)} \quad (u \in H_b^l(Q) : u|_{\Upsilon} = \psi).$$

We assume that $l + 2m - m_{i\mu} - 1 \geq 0$ for all i, μ and introduce the following bounded operator corresponding to nonlocal problem (1.2), (1.3):

$$\mathbf{L} = \{\mathbf{P}(x, D), \mathbf{B}_{i\mu}(x, D)\} : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \Upsilon) = H_b^l(G) \times \prod_{i=1}^{N_0} \prod_{\mu=1}^m H_b^{l+2m-m_{i\mu}-1/2}(\Upsilon_i).$$

From now on (unless the contrary is specified), we suppose that $b > l + 2m - 1$.

Let us explain the restriction on the exponent b . Suppose that the transformation ω_{is} takes a point $g \in \bar{\Upsilon}_i \cap \mathcal{K}_1$ to the point $\omega_{is}(g)$ so that $\omega_{is}(g) \in \mathcal{K}_2$ or $\omega_{is}(g) \in \mathcal{K}_3$. Since the function $u(x)$ belongs to the Sobolev space W_2^{l+2m} near the point $\omega_{is}(g)$, the function $u(\omega_{is}(x))$ belongs to the Sobolev space W_2^{l+2m} near the point g . However, if $b \leq l + 2m - 1$, the function $u(\omega_{is}(x))$ does not belong (in general) to the weighted space H_b^{l+2m} . Therefore, the trace $(B_{i\mu s}(x, D)u)(\omega_{is}(x))|_{\Upsilon_i}$ may not belong to the weighted space $H_b^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$, so the operator \mathbf{L} is not well defined. But if $b > l + 2m - 1$, then, by virtue of Lemma 5.2 [12], $W_2^{l+2m}(G) \subset H_b^{l+2m}(G)$. Thus, in this case, the operator \mathbf{L} is well defined.

Notice that, in two-dimensional case, problem (1.2), (1.3) can be considered in weighted spaces with arbitrary exponent b (see [9]). To this end, one should impose some consistency conditions (generated by the transformations ω_{is}); namely, one must assume that the solutions u as well as the right-hand side $\{f_0, g_{i\mu}\}$ belong to the corresponding weighted spaces not only near the set \mathcal{K}_1 but also near \mathcal{K}_2 and \mathcal{K}_3 . On the one hand, this situation is in detail considered in [9] (where the problems with transformations linear near \mathcal{K}_1 are studied). On the other hand, the changes described have nothing to do with the transformations ω_{is} near \mathcal{K}_1 . So, in two-dimensional case, we will omit the proofs of corresponding results concerning arbitrary values of b (see the end of § 5).

2. Now we consider the structure of the transformations ω_{is} near the set \mathcal{K}_1 in more detail. We denote by ω_{is}^{+1} the transformation $\omega_{is} : \mathcal{O}_i \rightarrow \omega_{is}(\mathcal{O}_i)$ and by $\omega_{is}^{-1} : \omega_{is}(\mathcal{O}_i) \rightarrow \mathcal{O}_i$ the transformation being inverse to ω_{is} . Consider a point $g \in \mathcal{K}_1$. The set of all points $\omega_{ip s_p}^{\pm 1}(\dots \omega_{i_1 s_1}^{\pm 1}(g)) \in \mathcal{K}_1$ ($1 \leq s_j \leq S_{ij}$, $j = 1, \dots, p$) (that is, points which can be obtained by consecutive applying to the point g the transformations $\omega_{ij s_j}^{+1}$ or $\omega_{ij s_j}^{-1}$ taking the points from \mathcal{K}_1 to those from \mathcal{K}_1) is called an *orbit* of $g \in \mathcal{K}_1$ and denoted by $\text{Orb}(g)$.

We introduce the set $\mathcal{S}_{i1} = \{0 \leq s \leq S_i : \omega_{is}(\bar{\Upsilon}_i) \cap \mathcal{K}_1 \neq \emptyset\}$. Evidently, $0 \in \mathcal{S}_{i1}$. Let the following conditions hold.

Condition 1.3. For each $g \in \mathcal{K}_1$

- (a) the set $\text{Orb}(g)$ consists of finitely many points g^j ($j = 1, \dots, N = N(g)$);
- (b) for the points g^j , there are neighborhoods

$$\hat{\mathcal{V}}(g^j) \subset \mathcal{V}(g^j) \subset \mathbb{R}^n \setminus \left\{ \bigcup_{i,s} \omega_{is}(\bar{\Upsilon}_i) \cup \mathcal{K}_2 \cup \mathcal{K}_3 \right\} \quad (s \notin \mathcal{S}_{i1})$$

such that (I) $\mathcal{V}(g^j) \cap \mathcal{V}(g^k) = \emptyset$ ($j \neq k$) and (II) if $g^j \in \bar{\Upsilon}_i$ and $\omega_{is}(g^j) = g^k$, then $\mathcal{V}(g^j) \subset \mathcal{O}_i$ and $\omega_{is}(\hat{\mathcal{V}}(g^j)) \subset \mathcal{V}(g^k)$.

Condition 1.4. For each $g \in \mathcal{K}_1$ and $j = 1, \dots, N(g)$, there is a non-degenerate smooth transformation $x \mapsto x'(g, j)$ mapping $\mathcal{V}(g^j)$ ($\hat{\mathcal{V}}(g^j)$) onto a neighborhood of the origin $\mathcal{V}_j(0)$ ($\hat{\mathcal{V}}_j(0)$) so that

- (a) the images of the sets $G \cap \mathcal{V}(g^j)$ ($G \cap \hat{\mathcal{V}}(g^j)$) and $\Upsilon_i \cap \mathcal{V}(g^j)$ ($\Upsilon_i \cap \hat{\mathcal{V}}(g^j)$) are respectively the intersection of the dihedral angle $\Omega_j = \{x = (y, z) \in \mathbb{R}^n : 0 < b'_j < \varphi < b''_j < 2\pi, z \in \mathbb{R}^{n-2}\}$ with $\mathcal{V}_j(0)$ ($\hat{\mathcal{V}}_j(0)$) and the intersection of the side of the angle Ω_j with $\mathcal{V}_j(0)$ ($\hat{\mathcal{V}}_j(0)$);
- (b) for $x \in \hat{\mathcal{V}}(g^j)$, the transformation $\omega_{is}(x)$ ($s \in \mathcal{S}_{i1} \setminus \{0\}$) in new coordinates has the form $(y', z') \mapsto (\omega'_{is}(y', z'), z')$, where $\omega'_{is}(y', z') = \mathcal{G}'_{is}y' + o(|x'|)$ with \mathcal{G}'_{is} being the operator of rotation by an angle φ'_{is} and expansion $\chi'_{is} > 0$ times in y' -plane; moreover, we assume that $\omega'_{is}(0, z) \equiv 0$;
- (c) in new coordinates, the operator \mathcal{G}'_{is} maps the side of the corresponding angle Ω_j ($j = j(i)$) onto an $(n-1)$ -dimensional half-plane being strictly inside an angle Ω_k ($k = k(i, s)$ and j can be different).

Conditions 1.3 and 1.4 are analogous to those in [9, 11], where the transformations linear near \mathcal{K}_1 (and arbitrary outside a neighborhood of \mathcal{K}_1) are studied.

Condition 1.3 (a) is in a sense analogous to Carleman's condition [4], which is used in the theory of nonlocal problems with transformations mapping the boundary of domain onto itself.

Condition 1.4, in particular, means that if $g \in \omega_{is}(\tilde{\Upsilon}_i \setminus \Upsilon_i) \cap \tilde{\Upsilon}_j \cap \mathcal{K}_1 \neq \emptyset$, then the surfaces $\omega_{is}(\tilde{\Upsilon}_i)$ and $\tilde{\Upsilon}_j$ have different tangent planes at the point g . The requirement that $\omega'_{is}(0, z) \equiv 0$ is necessary for representation (1.1) to be possible. If $\omega_{is}(\tilde{\Upsilon}_i \setminus \Upsilon_i) \subset \bar{G} \setminus \mathcal{K}_1$, then, like in [9, 11], we have no restrictions on a geometrical structure of $\omega_{is}(\tilde{\Upsilon}_i)$ near ∂G .

Remark 1.1. One can consider the more general case where, for $x \in \hat{\mathcal{V}}(g^j)$, the transformation $\omega_{is}(x)$ ($s \in \mathcal{S}_{i1} \setminus \{0\}$) in new coordinates has the form $(y', z') \mapsto (\omega'_{is}(y', z'), \omega''_{is}(y', z'))$, where $\omega'_{is}(y', z')$ is the same as before, $\omega''_{is}(y', z') = z' + o(|x'|)$, $\omega''_{is}(0, z') \equiv z'$ (the latter guarantees that item (a) in Condition 1.3 holds). However, for simplicity, we study the transformations described in Condition 1.4.

3. Let us write model problems corresponding to the points of \mathcal{K}_1 .

We fix a point $g \in \mathcal{K}_1$. Let $\text{supp } u \subset \left(\bigcup_{j=1}^{N(g)} \hat{\mathcal{V}}(g^j) \right) \cap \bar{G}$. We denote the function $u(x)$ for $x \in \mathcal{V}(g^j) \cap G$ by $u_j(x)$. If $g^j \in \tilde{\Upsilon}_i$, $x \in \hat{\mathcal{V}}(g^j)$, $\omega_{is}(x) \in \mathcal{V}(g^k)$, then we denote $u(\omega_{is}(x))$ by $u_k(\omega_{is}(x))$. Clearly, $u(\omega_{i0}(x)) \equiv u(x) \equiv u_j(x)$. Now nonlocal problem (1.2), (1.3) assumes the form

$$\begin{aligned} \mathbf{P}(x, D)u_j &= f_0(x) \quad (x \in \hat{\mathcal{V}}(g^j) \cap G), \\ \sum_{s \in \mathcal{S}_{i1}} (B_{i\mu s}(x, D)u_k)(\omega_{is}(x))|_{\Upsilon_i} &= g_{i\mu}(x) \\ (x \in \hat{\mathcal{V}}(g^j) \cap \Upsilon_i; i \in \{1 \leq i \leq N_0 : \hat{\mathcal{V}}(g^j) \cap \Upsilon_i \neq \emptyset\}; \\ j &= 1, \dots, N = N(g); \mu = 1, \dots, m). \end{aligned}$$

By virtue of Condition 1.4, in new coordinates the linear part \mathcal{G}'_{is} of the transformation ω'_{is} maps one of the sides of Ω_j ($j = j(i)$) onto an $(n-1)$ -dimensional half-plane being strictly inside Ω_k ($k = k(i, s)$ and j can be different). We denote all these $(n-1)$ -dimensional half-planes by $\Gamma_{k2}, \dots, \Gamma_{k,R_k} \subset \Omega_k$. (If none of the sides of the angles $\Omega_1, \dots, \Omega_N$ is mapped inside Ω_k , we put $R_k = 1$.) We also denote $b_{k1} = b'_k$, $b_{k,R_k+1} = b''_k$. Then the sets

$$\Gamma_{k\sigma} = \{x = (y, z) \in \mathbb{R}^n : \varphi = b_{k\sigma}, z \in \mathbb{R}^{n-2}\} \quad (\sigma = 1, R_k + 1)$$

are the sides of Ω_k , while the half-planes Γ_{kq} have the forms

$$\Gamma_{kq} = \{x = (y, z) \in \mathbb{R}^n : \varphi = b_{kq}, z \in \mathbb{R}^{n-2}\} \quad (q = 2, \dots, R_k),$$

where $0 < b_{k1} < \dots < b_{k, R_k+1} < 2\pi$.

Let us introduce the function $U_j(x') = u_j(x(x'))$ and denote x' again by x . Then, by virtue of Conditions 1.3 and 1.4, problem (1.2), (1.3) eventually assumes the form

$$\mathcal{P}_j(x, D_y, D_z)U_j = f_j(x) \quad (x \in \Omega_j), \quad (1.4)$$

$$\begin{aligned} & \mathcal{B}_{j\sigma\mu}(x, D_y, D_z)U \equiv B_{j\sigma\mu}(x, D_y, D_z)U_j|_{\Gamma_{j\sigma}} + \\ & + \sum_{k,q,s} (B_{j\sigma\mu kqs}(x, D_y, D_z)U_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} = g_{j\sigma\mu}(x) \quad (x \in \Gamma_{j\sigma}). \end{aligned} \quad (1.5)$$

Here (and further, until the contrary is indicated) $j, k = 1, \dots, N; \sigma = 1, R_j+1; q = 2, \dots, R_k; \mu = 1, \dots, m; s = 1, \dots, S_{j\sigma kq}; \mathcal{P}_j(x, D_y, D_z), B_{j\sigma\mu}(x, D_y, D_z)$, and $B_{j\sigma\mu kqs}(x, D_y, D_z)$ are operators of order $2m, m_{j\sigma\mu}$, and $m_{j\sigma\mu}$ respectively with variable C^∞ -coefficients; $\omega'_{j\sigma kqs}(y, z) = \mathcal{G}_{j\sigma kqs}y + o(|x|)$ with $\mathcal{G}_{j\sigma kqs}$ being the operator of rotation by an angle $\varphi_{j\sigma kq}$ and expansion $\chi_{j\sigma kqs} > 0$ times in y -plane; furthermore, $\omega'_{j\sigma kqs}(0, z) \equiv 0, b_{k1} < b_{j\sigma} + \varphi_{j\sigma kq} = b_{kq} < b_{k, R_k+1}$.

Let us define the spaces of vector-functions:

$$\begin{aligned} H_b^{l+2m, N}(\Omega) &= \prod_j H_b^{l+2m}(\Omega_j), \quad \mathcal{H}_b^{l, N}(\Omega, \Gamma) = \prod_j \mathcal{H}_b^l(\Omega_j, \Gamma_j), \\ \mathcal{H}_b^l(\Omega_j, \Gamma_j) &= H_b^l(\Omega_j) \times \prod_{\sigma, \mu} H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma}). \end{aligned}$$

We introduce the bounded operators

$$\mathcal{L}^\omega = \{\mathcal{P}_j(D_y, D_z), \mathcal{B}_{j\sigma\mu}^\omega(D_y, D_z)\} : H_b^{l+2m, N}(\Omega) \rightarrow \mathcal{H}_b^{l, N}(\Omega, \Gamma),$$

$$\mathcal{L}^\mathcal{G} = \{\mathcal{P}_j(D_y, D_z), \mathcal{B}_{j\sigma\mu}^\mathcal{G}(D_y, D_z)\} : H_b^{l+2m, N}(\Omega) \rightarrow \mathcal{H}_b^{l, N}(\Omega, \Gamma).$$

Here²

$$\mathcal{B}_{j\sigma\mu}^\omega(D_y, D_z)U = B_{j\sigma\mu}(D_y, D_z)U_j|_{\Gamma_{j\sigma}} + \sum_{k,q,s} (B_{j\sigma\mu kqs}(D_y, D_z)U_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}},$$

$$\mathcal{B}_{j\sigma\mu}^\mathcal{G}(D_y, D_z)U = B_{j\sigma\mu}(D_y, D_z)U_j|_{\Gamma_{j\sigma}} + \sum_{k,q,s} (B_{j\sigma\mu kqs}(D_y, D_z)U_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}}$$

with $\mathcal{P}_j(D_y, D_z), B_{j\sigma\mu}(D_y, D_z)$, and $B_{j\sigma\mu kqs}(D_y, D_z)$ being the principal homogeneous parts of the operators $\mathcal{P}_j(0, D_y, D_z), B_{j\sigma\mu}(0, D_y, D_z)$, and $B_{j\sigma\mu kqs}(0, D_y, D_z)$ respectively.

In what follows, we will write, for short, $\mathcal{P}_j, B_{j\sigma\mu}, B_{j\sigma\mu kqs}, \mathcal{B}_{j\sigma\mu}^\omega$, and $\mathcal{B}_{j\sigma\mu}^\mathcal{G}$ instead of $\mathcal{P}_j(D_y, D_z), B_{j\sigma\mu}(D_y, D_z), B_{j\sigma\mu kqs}(D_y, D_z), \mathcal{B}_{j\sigma\mu}^\omega(D_y, D_z)$, and $\mathcal{B}_{j\sigma\mu}^\mathcal{G}(D_y, D_z)$ respectively.

Notice that the operator $\mathcal{B}_{j\sigma\mu}^\omega$ contains nonlocal terms with nonlinear transformations $\omega'_{j\sigma kqs}$ while the operator $\mathcal{B}_{j\sigma\mu}^\mathcal{G}$ with linear ones $\mathcal{G}_{j\sigma kqs}$. Thus, the operators \mathcal{L}^ω and $\mathcal{L}^\mathcal{G}$ correspond to model problems with nonlinear and linearized transformations respectively.

²In what follows, we consider functions U_k with compact supports concentrated in a neighborhood of the origin and such that $(\omega'_{j\sigma kqs}(y, z), z) \in \Omega_k$ for $x \in \text{supp } U_k$. This guarantees that the operators $\mathcal{B}_{j\sigma\mu}^\omega(D_y, D_z)$ are well defined.

As we mentioned before, the problem with transformations linear near \mathcal{K}_1 was studied in [9, 10, 11]. In particular, its Fredholm solvability was investigated. In § 2 of the present paper, we will show that the operator \mathcal{L}^ω is neither a small nor compact perturbation of $\mathcal{L}^\mathcal{G}$ even if the functions U with arbitrary small supports are considered. That is why, to prove the Fredholm solvability of problem (1.2), (1.3) with nonlinear transformations, we have to obtain anew a priori estimates and construct a right regularizer (see §§ 4, 5).

4. Obtaining a priori estimates and constructing the right regularizer for problem (1.2), (1.3) will be based on the invertibility of the model operators $\mathcal{L}^\mathcal{G}$. Let us formulate the conditions under which the operator $\mathcal{L}^\mathcal{G}$ is an isomorphism. If $n \geq 3$, then, parallel to the operator in dihedral angles, we consider a model operator with parameter θ in plain angles. For any angle $K = \{y \in \mathbb{R}^2 : 0 < b' < \varphi < b'' < 2\pi\}$, we introduce the space $E_b^l(K)$ as a completion of $C_0^\infty(\bar{K} \setminus \{0\})$ with respect to the norm

$$\|u\|_{E_b^l(K)} = \left(\sum_{|\alpha| \leq l} \int_K |y|^{2b} (|y|^{2(|\alpha|-l)} + 1) |D_y^\alpha u(y)|^2 dy \right)^{1/2}.$$

For $l \geq 1$, we denote by $E_b^{l-1/2}(\gamma)$ the space of traces on a ray $\gamma \subset \bar{K}$ with the norm

$$\|\psi\|_{E_b^{l-1/2}(\gamma)} = \inf \|u\|_{E_b^l(K)} \quad (u \in E_b^l(K) : u|_\gamma = \psi).$$

One can find the constructive definitions of the trace spaces $H_b^{l-1/2}(\Upsilon)$ and $E_b^{l-1/2}(\gamma)$, equivalent to the above, in [25, § 1].

We introduce the spaces of vector-functions

$$\begin{aligned} E_b^{l+2m,N}(K) &= \prod_j E_b^{l+2m}(K_j), \quad \mathcal{E}_b^{l,N}(K, \gamma) = \prod_j \mathcal{E}_b^l(K_j, \gamma_j), \\ \mathcal{E}_b^l(K_j, \gamma_j) &= E_b^l(K_j) \times \prod_{\sigma, \mu} E_b^{l+2m-m_{j\sigma\mu}-1/2}(\gamma_{j\sigma}), \end{aligned}$$

where $K_j = \{y \in \mathbb{R}^2 : b_{j1} < \varphi < b_{j,R_j+1}\}$, $\gamma_{j\sigma} = \{y \in \mathbb{R}^2 : \varphi = b_{j\sigma}\}$.

We consider the bounded operator

$$\mathcal{L}^\mathcal{G}(\theta) = \{\mathcal{P}_j(D_y, \theta), \mathcal{B}_{j\sigma\mu}^\mathcal{G}(D_y, \theta)\} : E_b^{l+2m,N}(K) \rightarrow \mathcal{E}_b^{l,N}(K, \gamma),$$

where θ is an arbitrary point of the unit sphere $S^{n-3} = \{\theta \in \mathbb{R}^{n-2} : |\theta| = 1\}$.

5. Let us write the operators $\mathcal{P}_j(D_y, 0)$, $B_{j\sigma\mu}(D_y, 0)$, $B_{j\sigma\mu kqs}(D_y, 0)$ in polar coordinates: $\mathcal{P}_j(D_y, 0) = r^{-2m} \tilde{\mathcal{P}}_j(\varphi, D_\varphi, rD_r)$, $B_{j\sigma\mu}(D_y, 0) = r^{-m_{j\sigma\mu}} \tilde{B}_{j\sigma\mu}(\varphi, D_\varphi, rD_r)$, $B_{j\sigma\mu kqs}(D_y, 0) = r^{-m_{j\sigma\mu}} \tilde{B}_{j\sigma\mu kqs}(\varphi, D_\varphi, rD_r)$, where $D_\varphi = -i \frac{\partial}{\partial \varphi}$, $D_r = -i \frac{\partial}{\partial r}$. We consider the analytic operator-valued function $\tilde{\mathcal{L}}(\lambda) : W_2^{l+2m,N}(b_1, b_2) \rightarrow \mathcal{W}_2^{l,N}[b_1, b_2]$ given by

$$\begin{aligned} \tilde{\mathcal{L}}(\lambda) \tilde{U} &= \{\tilde{\mathcal{P}}_j(\varphi, D_\varphi, \lambda) \tilde{U}_j, \tilde{B}_{j\sigma\mu}(\varphi, D_\varphi, \lambda) \tilde{U}_j(\varphi)|_{\varphi=b_{j\sigma}} + \\ &\quad + \sum_{k,q,s} e^{(i\lambda-m_{j\sigma\mu}) \ln \chi_{j\sigma kqs}} \tilde{B}_{j\sigma\mu kqs}(\varphi, D_\varphi, \lambda) \tilde{U}_k(\varphi + \varphi_{j\sigma kqs})|_{\varphi=b_{j\sigma}}\}, \end{aligned}$$

where

$$W_2^{l+2m, N}(b_1, b_2) = \prod_j W_2^{l+2m}(b_{j1}, b_{j, R_j+1}), \quad \mathcal{W}_2^{l, N}[b_1, b_2] = \prod_j \mathcal{W}_2^l[b_{j1}, b_{j, R_j+1}],$$

$$\mathcal{W}_2^l[b_{j1}, b_{j, R_j+1}] = W_2^l(b_{j1}, b_{j, R_j+1}) \times \mathbb{C}^{2m}.$$

By virtue of Lemmas 2.1, 2.2 [10], there exists a finite-meromorphic operator-valued function $(\tilde{\mathcal{L}}^g)^{-1}(\lambda)$ such that $(\tilde{\mathcal{L}}^g)^{-1}(\lambda)$ is the inverse to $\tilde{\mathcal{L}}^g(\lambda)$ if λ is not a pole of $(\tilde{\mathcal{L}}^g)^{-1}(\lambda)$; furthermore, for every pole λ_0 , there is a $\delta > 0$ such that the set $\{\lambda \in \mathbb{C} : 0 < |\operatorname{Im} \lambda - \operatorname{Im} \lambda_0| < \delta\}$ contains no poles of $(\tilde{\mathcal{L}}^g)^{-1}(\lambda)$.

If $n = 2$, then, by Theorem 2.1 [10], the operator \mathcal{L}^g is an isomorphism if and only if the line $\operatorname{Im} \lambda = b + 1 - l - 2m$ contains no poles of $(\tilde{\mathcal{L}}^g)^{-1}(\lambda)$.

Suppose that $n \geq 3$ and assume that the system $\{B_{j\sigma\mu}(D_y, D_z)\}_{\mu=1}^m$ is normal on $\Gamma_{j\sigma}$ and the orders $m_{j\sigma\mu}$ of the operators $B_{j\sigma\mu}(D_y, D_z)$, $B_{j\sigma\mu kqs}(D_y, D_z)$ are less or equal to $2m - 1$. In this case, by virtue of Theorem 9.1 [13], the operator $\mathcal{L}^g(\theta)$ is Fredholm if and only if the line $\operatorname{Im} \lambda = b + 1 - l - 2m$ contains no poles of $(\tilde{\mathcal{L}}^g)^{-1}(\lambda)$. By Theorem 3.3 [10], if, in addition, $\dim \ker(\mathcal{L}^g(\theta)) = \operatorname{codim} \mathcal{R}(\mathcal{L}^g(\theta)) = 0$ for b replaced by $b - l$, l replaced by 0, and all $\theta \in S^{n-3}$, then the operator

$$\mathcal{L}^g = \{\mathcal{P}_j(D_y, D_z), \mathcal{B}_{j\sigma\mu}^g(D_y, D_z)\} : H_b^{l+2m, N}(\Omega) \rightarrow \mathcal{H}_b^{l, N}(\Omega, \Gamma)$$

is an isomorphism (see the corresponding example in [13, § 10]). Notice that if \mathcal{L}^g is not an isomorphism, then $\mathcal{L}^g(\theta)$ is not Fredholm (see Theorem 9.3 [13]).

Since the operators \mathcal{L}^ω , \mathcal{L}^g , $\mathcal{L}^g(\theta)$, and $\tilde{\mathcal{L}}^g(\lambda)$ corresponding to problem (1.4), (1.5) depend on the choice of $g \in \mathcal{K}_1$, we denote them by \mathcal{L}_g^ω , \mathcal{L}_g^g , $\mathcal{L}_g^g(\theta)$, and $\tilde{\mathcal{L}}_g^g(\lambda)$ respectively.

2 Example of nonlocal problem with nonlinear argument transformations

In this section, we show on a simple example that a problem with a transformation nonlinear in a neighborhood of \mathcal{K}_1 is neither a small nor compact perturbation of the problem with the linearized transformation.

1. Let us assume for simplicity that problem (1.2), (1.3) is considered in a plain domain. Let the model problem (1.4), (1.5) corresponding to some point of \mathcal{K}_1 have the form

$$\begin{aligned} \Delta u &= f(y) & (y \in K), \\ u|_{\gamma_1} + u(\omega'(y))|_{\gamma_1} &= g_1(y) & (y \in \gamma_1), \\ u|_{\gamma_2} &= g_2(y) & (y \in \gamma_2). \end{aligned}$$

Here $K = \{y \in \mathbb{R}^2 : r > 0, |\varphi| < \pi/2\}$ is a plain angle (of opening π) with the sides $\gamma_i = \{y \in \mathbb{R}^2 : r > 0, \varphi = (-1)^i \pi/2\}$ ($i = 1, 2$). We suppose that $\omega'(y) = \mu(\mathcal{G}y)$, where \mathcal{G} is the operator of rotation by the angle $\pi/2$ mapping γ_1 onto a ray $\gamma = \{y \in \mathbb{R}^2 : r > 0, \varphi = 0\}$;

$$\mu : (y_1, y_2) \mapsto \left(\frac{y_1}{\sqrt{1 + y_1^2}}, y_2 + \frac{y_1^2}{\sqrt{1 + y_1^2}} \right)$$

is an infinitely differentiable transformation mapping γ onto the curve $\mu(\gamma)$, which is tangent to γ at the origin (see Fig. 2.1).

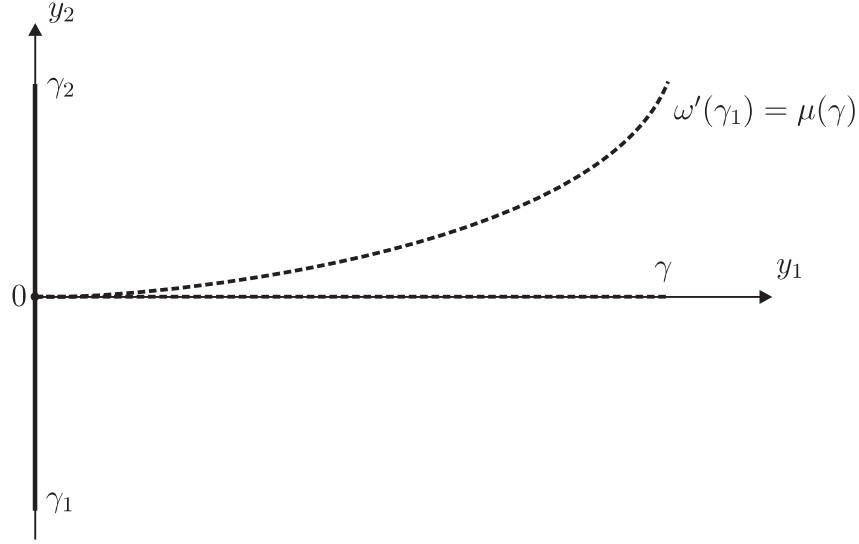


Figure 2.1: The angle K of opening π .

The operators $\mathcal{L}^\omega, \mathcal{L}^\mathcal{G} : H_b^{l+2}(K) \rightarrow H_b^l(K) \times \prod_{i=1}^2 H_b^{l+3/2}(\gamma_i)$ corresponding to the model problems with nonlinear and linearized transformations have the form

$$\mathcal{L}^\omega u = \{\Delta u, u|_{\gamma_1} + u(\omega'(y))|_{\gamma_1}, u|_{\gamma_2}\},$$

$$\mathcal{L}^\mathcal{G} u = \{\Delta u, u|_{\gamma_1} + u(\mathcal{G}y)|_{\gamma_1}, u|_{\gamma_2}\}.$$

Clearly, a non-zero component of the difference $\mathcal{L}^\mathcal{G} u - \mathcal{L}^\omega u$ is

$$u(\mathcal{G}y)|_{\gamma_1} - u(\omega'(y))|_{\gamma_1} = u(y)|_\gamma - u(\mu(y))|_\gamma.$$

We introduce the operator $A_\varepsilon : H_b^{l+2}(K) \rightarrow H_b^{l+3/2}(\gamma)$ with the domain $D(A_\varepsilon) = \{u \in H_b^{l+2}(K) : \text{supp } u \subset \{r < \varepsilon\} \cap \bar{K}\}$ given by

$$A_\varepsilon u(y) = u(y)|_\gamma - u(\mu(y))|_\gamma.$$

In this example, we prove that one cannot make the operator A_ε small or compact, choosing sufficiently small ε . For simplicity, we show this in the case where A_ε acts from $H_b^1(K)$ to $H_b^{1/2}(\gamma)$. The general case can be considered in the same way. We shall construct a sequence $u_\varepsilon \in D(A_\varepsilon)$, $\varepsilon \rightarrow 0$, such that

$$\|u_\varepsilon|_\gamma - u_\varepsilon(\mu(\cdot))|_\gamma\|_{H_b^{1/2}(\gamma)} \geq c \|u_\varepsilon\|_{H_b^1(K)},$$

where $c > 0$ is independent of ε .

Let us write the restriction of μ on γ in polar coordinates (φ, r) :

$$\mu|_\gamma : (0, r) \mapsto (\Phi(r), r),$$

where $\Phi(r) = \arctan r$. Clearly, $\Phi(0) = 0$, $\Phi(1) = \pi/4$, $\frac{1}{\sqrt{2}} \leq \frac{\Phi}{r}$, $\frac{d\Phi}{dr} \leq 1$ on $[0, 1]$.

Let us consider the transformation

$$\tilde{\mu} : (\varphi, r) \mapsto (\varphi + \Phi(r), r).$$

One can see that $u(\mu(y))|_\gamma = u(\tilde{\mu}(y))|_\gamma$ since $\mu|_\gamma = \tilde{\mu}|_\gamma$. Therefore, without loss of generality, we may assume that the transformation μ is given by

$$\mu : (\varphi, r) \mapsto (\varphi + \Phi(r), r).$$

Notice that the norm of any function $u \in H_b^1(K)$ written in polar coordinates is equivalent to

$$\left(\sum_{|\alpha| \leq 1} \int_0^\infty \int_{-\pi/2}^{\pi/2} r^{2b-1} |(rD_r)^{\alpha_1} D_\varphi^{\alpha_2} u(\varphi, r)|^2 d\varphi dr \right)^{1/2}.$$

Set $r = e^{-t}$; then, in new coordinates, the transformation μ assumes the form

$$\mu : (\varphi, t) \mapsto (\varphi + \Phi(e^{-t}), t).$$

Putting $v(\varphi, t) = u(\varphi, e^{-t})$, we see that the norm $\|u\|_{H_b^1(K)}$ is equivalent to the norm

$$\|v\|_{W_{2,b}^1(Q)} = \left(\sum_{|\alpha| \leq 1} \int_{-\infty}^\infty \int_{-\pi/2}^{\pi/2} e^{-2bt} |D_t^{\alpha_1} D_\varphi^{\alpha_2} v(\varphi, t)|^2 d\varphi dt \right)^{1/2}, \quad (2.1)$$

where $Q = \{t \in \mathbb{R}, |\varphi| < \pi/2\}$ and $W_{2,b}^1(Q)$ is the space with norm (2.1). Evidently, $W_{2,0}^1(Q)$ coincides with the Sobolev space $W_2^1(Q)$.

Since the norms $\|v\|_{W_{2,b}^1(Q)}$ and $\|e^{-bt}v\|_{W_2^1(Q)}$ are equivalent, it suffices to study the case where $b = 0$. In what follows, we consider functions $v(\varphi, t)$ with the support being a subset of the strip $\{|\varphi| < \pi/2\}$. Putting $v = 0$ for $|\varphi| \geq \pi/2$, we obtain $\|v\|_{W_2^1(Q)} = \|v\|_{W_2^1(\mathbb{R}^2)}$.

Thus, our task is reduced to constructing a sequence $v_s \in W_2^1(\mathbb{R}^2)$ such that $\text{supp } v_s \subset \{t > 2s, |\varphi| < \pi/2\}$ and

$$\|v_s(0, t) - v_s(\Phi(e^{-t}), t)\|_{W_2^{1/2}(\mathbb{R})} \geq c \|v_s\|_{W_2^1(\mathbb{R}^2)},$$

where $c > 0$ is independent of s .

To this end, we pass from variables (φ, t) to (φ, τ) : we introduce the sets

$$Q_s = \left\{ |\theta| \leq \frac{\pi}{2}, 2s \leq \tau \leq 2s+1 \right\}, \quad s = 0, 1, 2, \dots,$$

and put

$$\varphi = F(\theta, \tau), \quad t = \tau. \quad (2.2)$$

Here $F(\theta, \tau) = \theta e^{2s} \Phi(e^{-\tau})$ for $(\theta, \tau) \in Q_s$, $s = 0, 1, 2, \dots$, and $F(\theta, \tau)$ is extended onto $\mathbb{R}^2 \setminus \bigcup_{s=0}^\infty Q_s$ so that the transformation (2.2) remains continuously differentiable with the Jacobian $\frac{\partial F}{\partial \theta}$ such that

$$0 < c_1 \leq \left| \frac{\partial F}{\partial \theta} \right| \leq c_2 \quad \text{on } \mathbb{R}^2. \quad (2.3)$$

Such an extension does exist: indeed,

$$\frac{\partial F}{\partial \theta} = e^{2s} \Phi(e^{-\tau}), \quad \frac{\partial F}{\partial \tau} = -\theta e^{-\tau+2s} \frac{d\Phi}{dr} \Big|_{r=e^{-\tau}}, \quad (\theta, \tau) \in Q_s;$$

therefore (by virtue of the above properties of Φ), in $\bigcup_{s=0}^{\infty} Q_s$ the function $F(\theta, \tau)$ is continuously differentiable with respect to θ and τ and inequalities (2.3) hold.

One easily sees that, under change of variables (2.2), the segment $Q_s \cap \{\theta = 0\}$ is an image of the corresponding segment of the line $\{\varphi = 0\}$. Furthermore, the transformation μ on Q_s has the form

$$\mu : (\theta, \tau) \mapsto (\theta + e^{-2s}, \tau), \quad (\theta, \tau) \in Q_s. \quad (2.4)$$

We consider functions $f, g \in C^\infty(\mathbb{R})$ such that $\text{supp } f \subset \{|\theta| < \frac{\pi}{2}\}$, $f(0) \neq f(1)$, $\text{supp } g \subset \{0 < \tau < 1\}$, $g(\tau) \not\equiv 0$ and define the sequence $w_s(\theta, \tau) = f_s(\theta)g_s(\tau)$, where

$$f_s(\theta) = f(\theta e^{2s}), \quad g_s(\tau) = g((\tau - 2s)e^{2s}), \quad s = 0, 1, 2, \dots$$

Clearly, $\text{supp } w_s \subset Q_s$ (see Fig. 2.2).

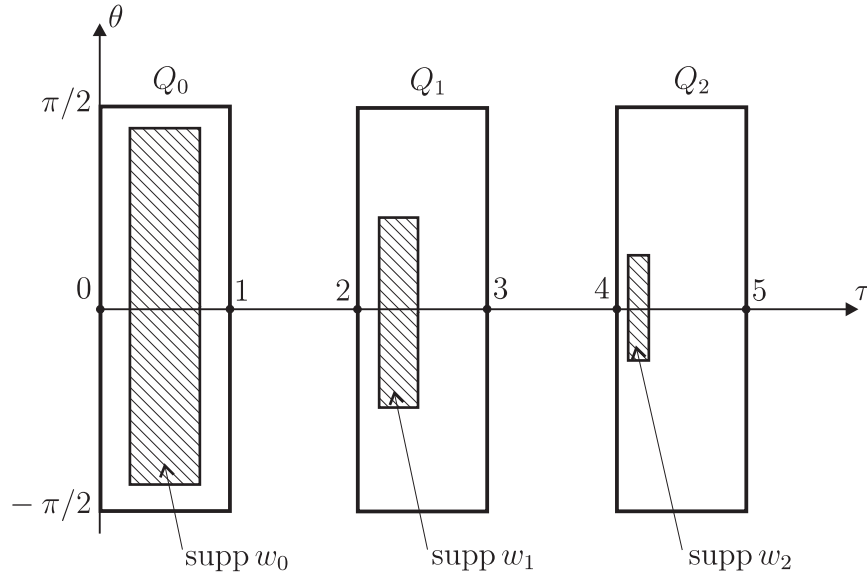


Figure 2.2: The supports of w_s are contained in the hatched domains.

We have

$$\begin{aligned} \|w_s\|_{W_2^1(\mathbb{R}^2)}^2 &= \|f_s\|_{L_2(\mathbb{R})}^2 \|g_s\|_{L_2(\mathbb{R})}^2 + \left\| \frac{df_s}{d\theta} \right\|_{L_2(\mathbb{R})}^2 \|g_s\|_{L_2(\mathbb{R})}^2 + \|f_s\|_{L_2(\mathbb{R})}^2 \left\| \frac{dg_s}{d\tau} \right\|_{L_2(\mathbb{R})}^2 = \\ &= e^{-4s} \|f\|_{L_2(\mathbb{R})}^2 \|g\|_{L_2(\mathbb{R})}^2 + \left\| \frac{df}{d\theta} \right\|_{L_2(\mathbb{R})}^2 \|g\|_{L_2(\mathbb{R})}^2 + \|f\|_{L_2(\mathbb{R})}^2 \left\| \frac{dg}{d\tau} \right\|_{L_2(\mathbb{R})}^2. \end{aligned} \quad (2.5)$$

Analogously, using the fact that the norm in $W_2^{1/2}(\mathbb{R})$ is given by

$$\|g\|_{W_2^{1/2}(\mathbb{R})} = \left(\|g\|_{L_2(\mathbb{R})}^2 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(\tau_1) - g(\tau_2)|^2}{|\tau_1 - \tau_2|^2} d\tau_1 d\tau_2 \right)^{1/2}$$

(see [26]) and the form (2.4) of the transformation μ in coordinates (θ, τ) , we get

$$\begin{aligned} \|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\|_{W_2^{1/2}(\mathbb{R})}^2 &= |f_s(0) - f_s(e^{-2s})|^2 \|g_s\|_{W_2^{1/2}(\mathbb{R})}^2 \geq \\ &\geq |f(0) - f(1)|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(\tau_1) - g(\tau_2)|^2}{|\tau_1 - \tau_2|^2} d\tau_1 d\tau_2. \end{aligned} \quad (2.6)$$

From (2.5) and (2.6), it follows that

$$\|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\|_{W_2^{1/2}(\mathbb{R})}^2 \geq c \|w_s\|_{W_2^1(\mathbb{R}^2)}^2.$$

2. Using the sequence w_s , one can easily show that, for any ε , the operator A_ε is not compact. Indeed, the sequence w_s is bounded in $W_2^1(\mathbb{R}^2)$. However, one cannot choose from $w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}$ a subsequence convergent in $W_2^{1/2}(\mathbb{R})$, since, according to (2.6), for all natural $s \neq h$ the expression

$$\begin{aligned} \|[w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}] - [w_h|_{\theta=0} - w_h(\mu(\cdot))|_{\theta=0}]\|_{W_2^{1/2}(\mathbb{R})} &= \\ &= \|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\|_{W_2^{1/2}(\mathbb{R})} + \|w_h|_{\theta=0} - w_h(\mu(\cdot))|_{\theta=0}\|_{W_2^{1/2}(\mathbb{R})} \end{aligned}$$

is bounded from below by a positive constant.

3 Argument transformations near the set \mathcal{K}_1

From the results of § 2, it follows that, to prove the Fredholm solvability of the problem with transformations nonlinear near \mathcal{K}_1 , one has to obtain anew a priori estimates and construct the right regularizer. To this end, we start by studying some properties of the transformations ω_{is} near the set \mathcal{K}_1 .

We fix a point $g \in \mathcal{K}_1$, make, for each $j = 1, \dots, N = N(g)$, the change of variables $x \mapsto x'(g, j)$, and consider the transformations $\omega'_{j\sigma k q s}(y, z)$ for $(y, z) \in \mathcal{V}_{\varepsilon_0}(0) = \{x \in \mathbb{R}^n : |x| < \varepsilon_0\}$. The number ε_0 is supposed to be small so that $\mathcal{V}_{\varepsilon_0}(0) \subset \hat{\mathcal{V}}_j(0)$, $j = 1, \dots, N$. In the sequel, we shall impose some additional conditions on ε_0 .

1. Before we proceed to study the transformations ω_{is} , let us prove an auxiliary result, which will be used for proving a lemma on a representation of ω_{is} in polar coordinates (see Lemma 3.2).

Lemma 3.1. *Let $h = h(r, z)$ be a function such that $|D_r^k D_z^\alpha h| \leq c_{k\alpha}$ for $r \geq 0$, $z \in \mathbb{R}^{n-2}$, $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$. Set $f(r, z) = r^{-l} h(r, z)$ for some $l \in \mathbb{N}$ and assume that $|f| \leq c$. Then $|D_r^k f| \leq c_k$ for $r \geq 0$, $z \in \mathbb{R}^{n-2}$, $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$, and any $k = 1, 2, \dots$*

Proof. 1) First, we consider the case where $l = 1$, that is $f(r, z) = r^{-1} h(r, z)$. By Leibnitz' formula, we have

$$\frac{\partial^k f(r, z)}{\partial r^k} = \sum_{s=0}^k \frac{(-1)^s k!}{(k-s)!} r^{-s-1} \frac{\partial^{k-s} h(r, z)}{\partial r^{k-s}}.$$

Expanding $\frac{\partial^{k-s}h}{\partial r^{k-s}}$ by the Taylor formula near $r = 0$ and using the boundedness of the derivatives of h , we obtain

$$\begin{aligned}\frac{\partial^k f(r, z)}{\partial r^k} &= \sum_{s=0}^k \frac{(-1)^s k!}{(k-s)!} r^{-s-1} \left[\sum_{p=0}^s \frac{1}{p!} \frac{\partial^{k-s+p} h}{\partial r^{k-s+p}}(0, z) r^p + \frac{\partial^{k+1} h}{\partial r^{k+1}}(\varkappa_{rz} r, z) r^{s+1} \right] = \\ &= \sum_{s=0}^k \sum_{p=0}^s \frac{(-1)^s k!}{(k-s)! p!} \frac{\partial^{k-s+p} h}{\partial r^{k-s+p}}(0, z) r^{-s-1+p} + O(1),\end{aligned}$$

where $\varkappa_{rz} \in (0, 1)$.

Putting $p' = s - p$ in the last sum and denoting p' again by p , we get

$$\frac{\partial^k f(r, z)}{\partial r^k} = \sum_{s=0}^k \sum_{p=0}^s \frac{(-1)^s k!}{(k-s)!(s-p)!} \frac{\partial^{k-p} h}{\partial r^{k-p}}(0, z) r^{-p-1} + O(1).$$

Write the coefficient $a_p(z)$ at r^{-p-1} on the right-hand side of the last identity:

$$\begin{aligned}a_p(z) &= \frac{\partial^{k-p} h}{\partial r^{k-p}}(0, z) \sum_{s=p}^k \frac{(-1)^s k!}{(k-s)!(s-p)!} = \\ &= \frac{\partial^{k-p} h}{\partial r^{k-p}}(0, z) (-1)^p \sum_{s=0}^{k-p} k(k-1) \cdots (k-(s+p)+1) \frac{1}{s!} (-1)^s, \quad p = 0, \dots, k.\end{aligned}$$

Since $|r^{-1}h(r, z)| \leq c$ by assumption, we have $h(0, z) \equiv 0$; therefore, $a_k(z) \equiv 0$. On the other hand, notice that, for $0 \leq p < k$, we have

$$\begin{aligned}0 &= \frac{d^p}{dt^p} (t+1)^k \Big|_{t=-1} = \left(\sum_{s=0}^{k-p} k(k-1) \cdots (k-(s+p)+1) \frac{1}{s!} t^s \right) \Big|_{t=-1} = \\ &= \sum_{s=0}^{k-p} k(k-1) \cdots (k-(s+p)+1) \frac{1}{s!} (-1)^s.\end{aligned}$$

Thus, $a_p(z) \equiv 0$ for all $p = 0, \dots, k$, and the lemma is proved for $l = 1$.

2) For $l \geq 2$, we use the mathematical induction method. Let the lemma be true for $l = 1, \dots, l_1 - 1$. We claim that it is true for $l = l_1$. We have $f = r^{-1}f_1$, where $f_1 = r^{-(l_1-1)}h$. Since $|f| \leq c$, it follows that $|f_1| \leq c$, and, therefore, by the inductive assumption (for $l = l_1 - 1$) the estimate $|D_r^k D_z^\alpha f_1| \leq c_{k\alpha}$ holds. Applying the inductive assumption once more (now, for $l = 1$), we get the conclusion of the lemma for $r^{-1}f_1$, that is, for $f = r^{-l_1}h$. \square

Now let us proceed to investigate the transformations ω_{is} . The following lemma describes the structure of $\omega'_{j\sigma kqs}$ in cylindrical coordinates. Such a representation turns out to be convenient for the study of nonlocal problems in weighted spaces.

Lemma 3.2. *For sufficiently small ε_0 , the transformation $\omega'_{j\sigma kqs}(y, z)|_{\Gamma_{j\sigma} \cap \mathcal{V}_{\varepsilon_0}(0)}$ can be represented in polar coordinates in the form*

$$(b_{j\sigma}, r) \mapsto (b_{kq} + \Phi_{j\sigma kqs}(r, z), \chi_{j\sigma kqs}r + R_{j\sigma kqs}(r, z)) \quad \text{for } (r^2 + |z|^2)^{1/2} \leq \varepsilon_0, \quad (3.1)$$

where $\Phi_{j\sigma kqs}(r, z)$, $R_{j\sigma kqs}(r, z)$ are infinitely differentiable functions such that

$$|\Phi_{j\sigma kqs}| \leq c\varepsilon_0, \quad |R_{j\sigma kqs}| \leq c\varepsilon_0 r, \quad (3.2)$$

$$|D_r^k D_z^\alpha \Phi_{j\sigma kqs}| \leq c_{k\alpha}, \quad |D_r^k D_z^\alpha (R_{j\sigma kqs}/r)| \leq c_{k\alpha}. \quad (3.3)$$

Here $k + |\alpha| \geq 1$; $c, c_{k\alpha} > 0$ are independent of ε_0 .

Proof. Let $\omega'_{j\sigma kqs}(y, z) = (\omega_{j\sigma kqs}^1(y, z), \omega_{j\sigma kqs}^2(y, z))$. By condition 1.4, we have $\omega_{j\sigma kqs}^i(0, z) \equiv 0$ ($i = 1, 2$); therefore, the Teylor formula near $r = 0$ implies

$$\omega_{j\sigma kqs}^i(r \cos b_{j\sigma}, r \sin b_{j\sigma}, z) = \left(\frac{\partial \omega_{j\sigma kqs}^i}{\partial y_1}(0, z) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^i}{\partial y_2}(0, z) \sin b_{j\sigma} \right) r + O(r^2). \quad (3.4)$$

Here $O(r^2)$ is a function with absolute values majorized by cr^2 , where c is independent of r and z . (To verify this, one should write the remainder of the Teylor formula in Lagrange's form and use smoothness of $\omega_{j\sigma kqs}^i$.) Expanding $\frac{\partial \omega_{j\sigma kqs}^i}{\partial y_1}(0, z)$ and $\frac{\partial \omega_{j\sigma kqs}^i}{\partial y_2}(0, z)$ by the Teylor formula near $z = 0$, from (3.4) we obtain

$$\omega_{j\sigma kqs}^i = \left(\frac{\partial \omega_{j\sigma kqs}^i}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^i}{\partial y_2}(0) \sin b_{j\sigma} \right) r + O(|z|)r + O(r^2). \quad (3.5)$$

Notice that $\frac{\partial \omega_{j\sigma kqs}^1}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^1}{\partial y_2}(0) \sin b_{j\sigma}$ and $\frac{\partial \omega_{j\sigma kqs}^2}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^2}{\partial y_2}(0) \sin b_{j\sigma}$ are not simultaneously equal to zero. (This follows from non-degeneracy of the Jacobian of the transformation $(y, z) \mapsto (\omega'_{j\sigma kqs}(y, z), z)$ at the origin.) For definiteness, we assume that

$$\frac{\partial \omega_{j\sigma kqs}^1}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^1}{\partial y_2}(0) \sin b_{j\sigma} \neq 0. \quad (3.6)$$

Hence, by virtue of (3.5),

$$\omega_{j\sigma kqs}^1 \neq 0 \quad \text{for } (r^2 + |z|^2)^{1/2} \leq \varepsilon_0 \quad (3.7)$$

with ε_0 small enough, and the transformation $\omega'_{j\sigma kqs}|_{\Gamma_{j\sigma} \cap \mathcal{V}_{\varepsilon_0}(0)}$ in polar coordinates has the form

$$(b_{j\sigma}, r) \mapsto \left(\arctan \frac{\omega_{j\sigma kqs}^2}{\omega_{j\sigma kqs}^1} + \pi l, \sqrt{\sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2} \right), \quad (3.8)$$

where $l = 0$ if $\omega_{j\sigma kqs}^1 > 0$ and $\omega_{j\sigma kqs}^2 \geq 0$, $l = 1$ if $\omega_{j\sigma kqs}^1 < 0$, $l = 2$ if $\omega_{j\sigma kqs}^1 > 0$ and $\omega_{j\sigma kqs}^2 < 0$.

From (3.5) and the Teylor formula, it follows that

$$\arctan \frac{\omega_{j\sigma kqs}^2}{\omega_{j\sigma kqs}^1} = \arctan \frac{\frac{\partial \omega_{j\sigma kqs}^2}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^2}{\partial y_2}(0) \sin b_{j\sigma}}{\frac{\partial \omega_{j\sigma kqs}^1}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^1}{\partial y_2}(0) \sin b_{j\sigma}} + O(|z|) + O(r),$$

$$\sqrt{\sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2} = r \sqrt{\sum_{i=1}^2 \left(\frac{\partial \omega_{j\sigma kqs}^i}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^i}{\partial y_2}(0) \sin b_{j\sigma} \right)^2} + O(|z|)r + O(r^2).$$

Setting

$$b_{kq} = \arctan \frac{\frac{\partial \omega_{j\sigma kqs}^2}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^2}{\partial y_2}(0) \sin b_{j\sigma}}{\frac{\partial \omega_{j\sigma kqs}^1}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^1}{\partial y_2}(0) \sin b_{j\sigma}} + \pi l,$$

$$\chi_{j\sigma kqs} = \sqrt{\sum_{i=1}^2 \left(\frac{\partial \omega_{j\sigma kqs}^i}{\partial y_1}(0) \cos b_{j\sigma} + \frac{\partial \omega_{j\sigma kqs}^i}{\partial y_2}(0) \sin b_{j\sigma} \right)^2},$$

we get formula (3.1) and inequalities (3.2).

Let us prove the first inequality in (3.3). By (3.7), we have $\left| \frac{\omega_{j\sigma kqs}^2}{\omega_{j\sigma kqs}^1} \right| \leq c$ for $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$. Therefore, by virtue of (3.1) and (3.8), it suffices to prove that the derivatives $D_r^k D_z^\alpha \frac{\omega_{j\sigma kqs}^2}{\omega_{j\sigma kqs}^1}$ are bounded. Clearly, we have

$$\frac{\omega_{j\sigma kqs}^2}{\omega_{j\sigma kqs}^1} = \frac{r^{-1} \omega_{j\sigma kqs}^2}{r^{-1} \omega_{j\sigma kqs}^1}.$$

From (3.5) and (3.6), it follows that $r^{-1} \omega_{j\sigma kqs}^1 \neq 0$ for $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$. Hence, it suffices to prove that

$$|D_r^k D_z^\alpha (r^{-1} \omega_{j\sigma kqs}^i)| = |D_r^k (r^{-1} D_z^\alpha \omega_{j\sigma kqs}^i)| \leq c_{k\alpha}, \quad i = 1, 2.$$

But the function $D_z^\alpha \omega_{j\sigma kqs}^i$ is infinitely differentiable for $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$; furthermore, since $\omega_{j\sigma kqs}^i(0, z) \equiv 0$, we have $D_z^\alpha \omega_{j\sigma kqs}^i = O(r)$. Therefore, $|r^{-1} D_z^\alpha \omega_{j\sigma kqs}^i| \leq c_\alpha$. Now the conclusion of the lemma follows from Lemma 3.1.

Similarly, one can prove the second inequality in (3.3). From (3.1) and (3.8), it follows that

$$\frac{R_{j\sigma kqs}(r, z)}{r} = \sqrt{\sum_{i=1}^2 \frac{(\omega_{j\sigma kqs}^i)^2}{r^2}} - \chi_{j\sigma kqs}.$$

By virtue of (3.5) and (3.6), we obtain $\sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2 / r^2 \neq 0$ for $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$; therefore, it suffices to prove that

$$\left| D_r^k D_z^\alpha \sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2 / r^2 \right| \leq c_{k\alpha}.$$

But the function $D_z^\alpha \sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2$ is infinitely differentiable for $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$; furthermore, since $\omega_{j\sigma kqs}^i(0, z) \equiv 0$, we have $D_z^\alpha \sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2 = O(r^2)$. Hence, $\left| D_z^\alpha \sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2 / r^2 \right| \leq c_\alpha$, and the conclusion of the lemma again follows from Lemma 3.1. \square

2. Denote $\delta = \min\{b_{j,q+1} - b_{jq}\}/2$ ($j = 1, \dots, N$; $q = 1, \dots, R_j$), $d_1 = \min\{1, \chi_{j\sigma kqs}\}/2$, $d_2 = 2 \max\{1, \chi_{j\sigma kqs}\}$. Let ε_0 be so small that

$$|\Phi_{j\sigma kqs}| \leq \delta/2, \quad |R_{j\sigma kqs}| \leq \chi_{j\sigma kqs} r/2 \quad \text{for } (r^2 + |z|^2)^{1/2} \leq \varepsilon_0/d_1. \quad (3.9)$$

The existence of such an ε_0 follows from Lemma 3.2.

We introduce infinitely differentiable functions $\zeta_{j\sigma,i}(\varphi)$, $\zeta_{kq,i}(\varphi)$ such that

$$\zeta_{j\sigma,i}(\varphi) = 1 \text{ for } |b_{j\sigma} - \varphi| \leq \delta/2^{i+1}, \quad \zeta_{j\sigma,i}(\varphi) = 0 \text{ for } |b_{j\sigma} - \varphi| \geq \delta/2^i, \quad (3.10)$$

$$\zeta_{kq,i}(\varphi) = \zeta_{j\sigma,i}(\varphi - \varphi_{j\sigma kq}),$$

$i = 0, \dots, 4$. Clearly, $\zeta_{kq,i}(\varphi) = 1$ for $|b_{kq} - \varphi| \leq \delta/2^{i+1}$, $\zeta_{kq,i}(\varphi) = 0$ for $|b_{kq} - \varphi| \geq \delta/2^i$.

Let us consider the transformation $\tilde{\omega}'_{j\sigma kqs}(y, z)$ that are given in polar coordinates by

$$(\varphi, r) \mapsto (\varphi + \varphi_{j\sigma kq} + \Phi_{j\sigma kqs}(r, z), \chi_{j\sigma kqs}r + R_{j\sigma kqs}(r, z)). \quad (3.11)$$

By virtue of Lemma 3.2, we have $\tilde{\omega}'_{j\sigma kqs}(y, z)|_{\Gamma_{j\sigma} \cap \mathcal{V}_{\varepsilon_0}(0)} = \omega'_{j\sigma kqs}(y, z)|_{\Gamma_{j\sigma} \cap \mathcal{V}_{\varepsilon_0}(0)}$; therefore, in what follows, we can assume that the transformation $\omega'_{j\sigma kqs}(y, z)$ is given by (3.11). Notice that now $\omega'_{j\sigma kqs}(y, z)$ may have (in general) a singularity at the origin, since the new transformation $\omega'_{j\sigma kqs}(y, z)$ coincides with the old one $\omega'_{j\sigma kqs}(y, z)$ only on $\Gamma_{j\sigma} \cap \mathcal{V}_{\varepsilon_0}(0)$.

For any function $W(y, z)$, we denote $\hat{W}(y, z) = W(\omega'_{j\sigma kqs}(\mathcal{G}_{j\sigma kqs}^{-1}y, z), z)$. By virtue of Lemma 3.2, $\omega'_{j\sigma kqs}(\mathcal{G}_{j\sigma kqs}^{-1}y, z)$ in polar coordinates has the form

$$(\varphi, r) \mapsto (\varphi + \Phi'_{j\sigma kqs}(r, z), r + R'_{j\sigma kqs}(r, z)), \quad (3.12)$$

where $\Phi'_{j\sigma kqs}(r, z) = \Phi_{j\sigma kqs}(\chi_{j\sigma kqs}^{-1}r, z)$, $R'_{j\sigma kqs}(r, z) = R_{j\sigma kqs}(\chi_{j\sigma kqs}^{-1}r, z)$. It is easy to see that $\Phi'_{j\sigma kqs}$ and $R'_{j\sigma kqs}$ also satisfy inequalities (3.2), (3.3).

Lemma 3.3. *For sufficiently small ε_0 and any $W \in H_b^l(\Omega_k)$ with $\text{supp } W \subset \bar{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0)$ we have $\zeta_{kq,1}\hat{W} \in H_b^l(\Omega_k)$ and*

$$\|\zeta_{kq,1}\hat{W}\|_{H_b^l(\Omega_k)} \leq c\|W\|_{H_b^l(\Omega_k)},$$

where $q = 2, \dots, R_k$; $c > 0$ is independent of W and ε_0 .

Proof. In the proof, we shall use the following obvious assertion:

$$W \in H_b^l(\Omega_k) \iff D^\alpha W \in H_{b+|\alpha|-l}^0(\Omega_k), \quad |\alpha| \leq l. \quad (3.13)$$

From formula (3.12) and inequalities (3.9), it follows that the transformation (3.12) maps $\overline{\mathcal{V}_{\varepsilon_0}(0)} \cap \{x : |\varphi - b_{kq}| < \delta\} \cap \Omega_k$ into Ω_k for $q = 2, \dots, R_k$. Furthermore, inequalities (3.2) and (3.3) imply that, for small ε_0 , the absolute value of the Jacobian of transformation (3.12) is bounded and does not vanish in $\overline{\mathcal{V}_{\varepsilon_0}(0)} \cap \{x : |\varphi - b_{kq}| < \delta\} \cap \Omega_k$. This proves the lemma for $l = 0$ and $\zeta_{kq,0}$ substituted for $\zeta_{kq,1}$.

Let us consider functions $\zeta_{kq,0}^p \in C_0^\infty(\mathbb{R})$ ($p = 0, \dots, l$) such that $\zeta_{kq,0}^0 = \zeta_{kq,0}$, $\zeta_{kq,0}^l = \zeta_{kq,1}$, and $\zeta_{kq,0}^{p-1}(\varphi) = 1$ for $\varphi \in \text{supp } \zeta_{kq,0}^p$ ($p = 1, \dots, l$). Let us assume that the lemma is true for $l = p - 1$ and $\zeta_{kq,0}^{p-1}$ substituted for $\zeta_{kq,1}$. We claim that it is true for $l = p$ and $\zeta_{kq,0}^p$ substituted for $\zeta_{kq,1}$ ($p \geq 1$). Indeed, let $W \in H_b^p(\Omega_k)$; then $\frac{1}{r} \frac{\partial W}{\partial \varphi}, \frac{\partial W}{\partial r}, \frac{\partial W}{\partial z_\xi} \in H_b^{p-1}(\Omega_k)$, $\xi = 1, \dots, n - 2$.

Therefore, by the inductive assumption, we have $\zeta_{kq,0}^{p-1} \frac{1}{r} \frac{\partial W}{\partial \varphi}, \zeta_{kq,0}^{p-1} \frac{\partial W}{\partial r}, \zeta_{kq,0}^{p-1} \frac{\partial W}{\partial z_\xi} \in H_b^{p-1}(\Omega_k)$. From this, relations

$$\begin{aligned} \frac{1}{r} \frac{\partial \hat{W}_k}{\partial \varphi} &= \widehat{\frac{1}{r} \frac{\partial W}{\partial \varphi}} \cdot (1 + \frac{R'_{j\sigma kqs}}{r}), \\ \frac{\partial \hat{W}_k}{\partial r} &= \widehat{\frac{1}{r} \frac{\partial W}{\partial \varphi}} \cdot (1 + \frac{R'_{j\sigma kqs}}{r}) \cdot r \frac{\partial \Phi'_{j\sigma kqs}}{\partial r} + \widehat{\frac{\partial W}{\partial r}} \cdot (1 + \frac{\partial R'_{j\sigma kqs}}{\partial r}), \\ \frac{\partial \hat{W}_k}{\partial z_\xi} &= \widehat{\frac{1}{r} \frac{\partial W}{\partial \varphi}} \cdot (1 + \frac{R'_{j\sigma kqs}}{r}) \cdot r \frac{\partial \Phi'_{j\sigma kqs}}{\partial z_\xi} + \widehat{\frac{\partial W}{\partial r}} \cdot \frac{\partial R'_{j\sigma kqs}}{\partial z_\xi} + \widehat{\frac{\partial W}{\partial z_\xi}}, \end{aligned} \quad (3.14)$$

inequalities (3.2), (3.3), and Lemma³ 2.1 [27], we get

$$\zeta_{kq,0}^{p-1} \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi}, \quad \zeta_{kq,0}^{p-1} \frac{\partial \hat{W}}{\partial r}, \quad \zeta_{kq,0}^{p-1} \frac{\partial \hat{W}}{\partial z_\xi} \in H_b^{p-1}(\Omega_k). \quad (3.15)$$

Furthermore, the relation $W \in H_b^p(\Omega_k)$, embedding $H_b^p(\Omega_k) \subset H_{b-p}^0(\Omega_k)$, and the conclusion of the lemma for $l = 0$ imply $\zeta_{kq,0}^p \hat{W} \in H_{b-p}^0(\Omega_k)$. From this, (3.13), and (3.15), it follows that $D^\alpha(\zeta_{kq,0}^p \hat{W}) \in H_{b+|\alpha|-p}^0(\Omega_k)$, $|\alpha| \leq p$. Once more using (3.13), we complete the proof. \square

Thus, we proved that the operator $W \mapsto \zeta_{kq,1} \hat{W}$ is bounded in $H_b^l(\Omega_k)$.

Lemma 3.4. *For any $W \in H_b^l(\Omega_k)$ with $\text{supp } W \subset \bar{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0)$ and any multi-index γ , $1 \leq |\gamma| \leq l$, the following inequality holds:*

$$\|\zeta_{kq,2} D^\gamma \hat{W} - \zeta_{kq,2} \widehat{D^\gamma W}\|_{H_b^{l-|\gamma|}(\Omega_k)} \leq c\varepsilon_0 \|W\|_{H_b^l(\Omega_k)}, \quad (3.16)$$

where $q = 2, \dots, R_k$; $c > 0$ is independent of W and ε_0 .

Proof. We introduce functions $\zeta_{kq,1}^p \in C_0^\infty(\mathbb{R})$ ($p = 1, \dots, l$) such that $\zeta_{kq,1}^1 = \zeta_{kq,1}$, $\zeta_{kq,1}^l = \zeta_{kq,2}$, and $\zeta_{kq,1}^{p-1}(\varphi) = 1$ for $\varphi \in \text{supp } \zeta_{kq,1}^p$ ($p = 2, \dots, l$).

Let $|\gamma| = 1$; then it suffices to prove inequality (3.16) for the case where the operator D^γ is replaced by $\frac{1}{r} \frac{\partial}{\partial \varphi}$, $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial z_\xi}$. Let us consider the operator $\frac{1}{r} \frac{\partial}{\partial \varphi}$ (the other operators can be considered in the same way). Combining the first relation in (3.14) with Leibniz' formula, we get

$$\begin{aligned} \left\| \zeta_{kq,1}^1 \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} - \zeta_{kq,1}^1 \widehat{\frac{1}{r} \frac{\partial W}{\partial \varphi}} \right\|_{H_b^{l-1}(\Omega_k)}^2 &= \left\| \zeta_{kq,1}^1 \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} \frac{R'_{j\sigma qks}}{r} \right\|_{H_b^{l-1}(\Omega_k)}^2 \leq \\ &\leq k_1 \sum_{|\alpha| \leq l-1} \sum_{|\beta| \leq |\alpha|} \int_{\bar{\Omega}_k} r^{2(b+|\alpha|-(l-1))} \left| D^{\alpha-\beta} \frac{R'_{j\sigma qks}}{r} \right|^2 \left| D^\beta \left(\zeta_{kq,1}^1 \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} \right) \right|^2 dx. \end{aligned}$$

From this, the last inequality in (3.2), and the last inequality in (3.3), we obtain

$$\left\| \zeta_{kq,1}^1 \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} - \zeta_{kq,1}^1 \widehat{\frac{1}{r} \frac{\partial W}{\partial \varphi}} \right\|_{H_b^{l-1}(\Omega_k)}^2 \leq k_2 \varepsilon_0^2 \left\| \zeta_{kq,1}^1 \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} \right\|_{H_b^{l-1}(\Omega_k)}^2. \quad (3.17)$$

Estimate (3.17) and Lemma 3.3 prove the lemma for $|\gamma| = 1$ and $\zeta_{kq,1}^1$ substituted for $\zeta_{kq,2}$.

We assume that the lemma is true for $1 \leq |\gamma| \leq p-1$ and $\zeta_{kq,1}^{p-1}$ substituted for $\zeta_{kq,2}$. Let us prove that it is true for $|\gamma| = p$ and $\zeta_{kq,1}^p$ substituted for $\zeta_{kq,2}$ ($p \geq 2$). We have

$$\begin{aligned} \|\zeta_{kq,1}^p D^\gamma \hat{W} - \zeta_{kq,1}^p \widehat{D^\gamma W}\|_{H_b^{l-|\gamma|}(\Omega_k)} &\leq \|\zeta_{kq,1}^p D^{|\gamma|-1} (D^1 \hat{W}) - \zeta_{kq,1}^p D^{|\gamma|-1} \widehat{D^1 W}\|_{H_b^{l-|\gamma|}(\Omega_k)} + \\ &+ \|\zeta_{kq,1}^p D^{|\gamma|-1} \widehat{D^1 W} - \zeta_{kq,1}^p D^{|\gamma|-1} (\widehat{D^1 W})\|_{H_b^{l-|\gamma|}(\Omega_k)} \leq k_3 (\|\zeta_{kq,1}^{p-1} D^1 \hat{W} - \zeta_{kq,1}^{p-1} \widehat{D^1 W}\|_{H_b^{l-1}(\Omega_k)} + \\ &+ \|\zeta_{kq,1}^p D^{|\gamma|-1} \widehat{D^1 W} - \zeta_{kq,1}^p D^{|\gamma|-1} (\widehat{D^1 W})\|_{H_b^{l-|\gamma|}(\Omega_k)}), \quad (3.18) \end{aligned}$$

³ Lemma 2.1 [27] (and Lemmas 2.2, 3.5, 3.6 [27], see below) is proved by Kondrat'ev for domains with angular or conical points. However, it is easy to see that it remains true for the domains with edges under consideration.

where $D^{|\gamma|-1}$ and D^1 are some derivatives of order $|\gamma| - 1$ and 1 respectively. By the inductive assumption, for each of the two norms on the right-hand side of (3.18), the following estimates hold:

$$\begin{aligned} & \|\zeta_{kq,1}^{p-1} D^1 \hat{W} - \zeta_{kq,1}^{p-1} \widehat{D^1 W}\|_{H_b^{l-1}(\Omega_k)} \leq k_4 \varepsilon_0 \|W\|_{H_b^l(\Omega_k)}, \\ & \|\zeta_{kq,1}^p D^{|\gamma|-1} \widehat{D^1 W} - \zeta_{kq,1}^p D^{|\gamma|-1} (\widehat{D^1 W})\|_{H_b^{l-|\gamma|}(\Omega_k)} \leq k_5 \varepsilon_0 \|D^1 W\|_{H_b^{l-1}(\Omega_k)} \leq k_6 \varepsilon_0 \|W\|_{H_b^l(\Omega_k)}. \end{aligned}$$

This and (3.18) imply the conclusion of the lemma. \square

Notice that the multiplier ε_0 appears in (3.16) since the minuend and subtrahend both contain the same transformation $\omega'_{j\sigma kqs}(\mathcal{G}_{j\sigma kqs}^{-1} y, z)$, but the minuend is the derivative D^γ of the transformed function \hat{W} while the subtrahend is the transformation of the derivative $D^\gamma W$.

Lemma 3.5. *For any $U_k \in H_b^{l+2m}(\Omega_k)$ with $\text{supp } U_k \subset \bar{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0)$, the following inequality holds:*

$$\begin{aligned} & \| (B_{j\sigma\mu kqs} U_k)(\mathcal{G}_{j\sigma kqs} y, z)|_{\Gamma_{j\sigma}} - (B_{j\sigma\mu kqs} U_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} \|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \leq \\ & \leq c(\varepsilon_0 \|U_k\|_{H_b^{l+2m}(\Omega_k)} + \|\zeta_{kq,3} U_k - \zeta_{kq,3} \hat{U}_k\|_{H_b^{l+2m}(\Omega_k)}), \end{aligned} \quad (3.19)$$

where $c > 0$ is independent of U and ε_0 .

Proof. Using the boundedness of the trace operator in weighted spaces, we get

$$\begin{aligned} & \| (B_{j\sigma\mu kqs} U_k)(\mathcal{G}_{j\sigma kqs} y, z)|_{\Gamma_{j\sigma}} - (B_{j\sigma\mu kqs} U_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} \|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \leq \\ & \leq k_1 \|\zeta_{kq,4} B_{j\sigma\mu kqs} U_k - \zeta_{kq,4} \widehat{B_{j\sigma\mu kqs} U_k}\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)} \leq \\ & \leq k_1 (\|\zeta_{kq,4} B_{j\sigma\mu kqs} U_k - \zeta_{kq,4} B_{j\sigma\mu kqs} \hat{U}_k\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)} + \\ & \quad + \|\zeta_{kq,4} B_{j\sigma\mu kqs} \hat{U}_k - \zeta_{kq,4} \widehat{B_{j\sigma\mu kqs} U_k}\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)}). \end{aligned} \quad (3.20)$$

Let us estimate the first norm on the right-hand side of (3.20) as follows:

$$\|\zeta_{kq,4} B_{j\sigma\mu kqs} U_k - \zeta_{kq,4} B_{j\sigma\mu kqs} \hat{U}_k\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)} \leq k_2 \|\zeta_{kq,3} U_k - \zeta_{kq,3} \hat{U}_k\|_{H_b^{l+2m}(\Omega_k)}. \quad (3.21)$$

The second norm on the right-hand side of (3.20) can be estimated with the help of Lemma 3.4:

$$\|\zeta_{kq,4} B_{j\sigma\mu kqs} \hat{U}_k - \zeta_{kq,4} \widehat{B_{j\sigma\mu kqs} U_k}\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)} \leq k_3 \varepsilon_0 \|U_k\|_{H_b^{l+2m}(\Omega_k)}. \quad (3.22)$$

From (3.20)–(3.22), the conclusion of the lemma follows. \square

Notice that the right-hand side of (3.19) contains the norm of the difference of the non-transformed function and the transformed one. To estimate such differences, we need the following result.

Lemma 3.6. *For any $W \in H_{b+1}^1(\Omega_k)$ with $\text{supp } W \subset \bar{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0)$, the following inequality holds:*

$$\|\zeta_{kq,1} W - \zeta_{kq,1} \hat{W}\|_{H_b^0(\Omega_k)} \leq c \varepsilon_0 \|W\|_{H_{b+1}^1(\Omega_k)}. \quad (3.23)$$

where $c > 0$ is independent of W and ε_0 .

Proof. Writing the arguments of the functions W and \hat{W} in cylindrical coordinates, we obtain

$$\begin{aligned} \|\zeta_{kq,1}W - \zeta_{kq,1}\hat{W}\|_{H_b^0(\Omega_k)} &\leq \|\zeta_{kq,1}W(\varphi, r, z) - \zeta_{kq,1}W(\varphi + \Phi'_{j\sigma kqs}(r, z), r, z)\|_{H_b^0(\Omega_k)} + \\ &\quad + \|\zeta_{kq,1}W(\varphi + \Phi'_{j\sigma kqs}(r, z), r, z) - \zeta_{kq,1}W(\varphi + \Phi'_{j\sigma kqs}(r, z), r + R'_{j\sigma kqs}(r, z), z)\|_{H_b^0(\Omega_k)}. \end{aligned} \quad (3.24)$$

Using the Schwartz inequality, we estimate the square of the first norm on the right-hand side of (3.24):

$$\begin{aligned} \|\zeta_{kq,1}W(\varphi, r, z) - \zeta_{kq,1}W(\varphi + \Phi'_{j\sigma kqs}(r, z), r, z)\|_{H_b^0(\Omega_k)}^2 &= \\ &= \int_{\mathbb{R}^{n-2}} dz \int_0^\infty r^{2b} r dr \int_{b_{k1}}^{b_{k2}} \left| \zeta_{kq,1} \int_{\varphi}^{\varphi + \Phi'_{j\sigma kqs}(r, z)} \frac{\partial W}{\partial \varphi'} d\varphi' \right|^2 d\varphi \leq \\ &\quad \int_{\mathbb{R}^{n-2}} dz \int_0^\infty r^{2b} r dr \int_{b_{k1}}^{b_{k2}} |\zeta_{kq,1}|^2 |\Phi'_{j\sigma kqs}(r, z)| \cdot \left| \int_{\varphi}^{\varphi + \Phi'_{j\sigma kqs}(r, z)} \left| \frac{\partial W}{\partial \varphi'} \right|^2 d\varphi' \right| d\varphi. \end{aligned}$$

Taking into account the restrictions on the support of the functions W and $\zeta_{kq,1}$ and inequalities (3.9), we can change the order of integration with respect to φ and φ' ; as a result, using (3.2), we get

$$\begin{aligned} \|\zeta_{kq,1}W(\varphi, r, z) - \zeta_{kq,1}W(\varphi + \Phi'_{j\sigma kqs}(r, z), r, z)\|_{H_b^0(\Omega_k)}^2 &\leq \\ &\leq k_1 \int_{\mathbb{R}^{n-2}} dz \int_0^\infty r^{2b} r |\Phi'_{j\sigma kqs}(r, z)|^2 dr \int_{b_{k1}}^{b_{k2}} \left| \frac{\partial W}{\partial \varphi} \right|^2 d\varphi \leq \\ &\leq k_2 \varepsilon_0^2 \int_{\mathbb{R}^{n-2}} dz \int_0^\infty r^{2(b+1)} r dr \int_{b_{k1}}^{b_{k2}} \left| \frac{1}{r} \frac{\partial W}{\partial \varphi} \right|^2 d\varphi \leq k_3 \varepsilon_0^2 \|W\|_{H_{b+1}^1(\Omega_k)}^2. \end{aligned}$$

Similarly, one can estimate the square of the second norm on the right-hand side of (3.24). \square

Thus, the multiplier ε_0 appears in (3.23) if one increases the order of differentiation by 1. (The left-hand side of (3.23) contains the norm in $H_b^0(\Omega_k)$ while the right-hand side does in $H_{b+1}^1(\Omega_k)$.) This can be explained as follows: unlike in (3.16), in this case one estimates the difference of the two functions the first one of which does not contain a transformation while the second one does.

4 A priori estimates of solutions

In this section, we prove an a priori estimate for the operator \mathbf{L} , which guarantees that its kernel is of finite dimension and its range is closed.

1. First, we prove an a priori estimate for functions with the support being a subset of some neighborhood of \mathcal{K}_1 . To this end, we will use the invertibility of the model operators \mathcal{L}_g^g , $g \in \mathcal{K}_1$, with linear transformations as well as Lemmas 3.3–3.6. Then, in subsection 2 of this section, using the results of [11] and Lemma 5.2 [12], we will obtain a priori estimates for functions with the support in the whole of \bar{G} .

We denote $\mathcal{O}_\varepsilon(\mathcal{K}_1) = \{x \in \mathbb{R}^n : \text{dist}(x, \mathcal{K}_1) < \varepsilon\}$.

Lemma 4.1. *Let Conditions 1.1–1.4 hold and, for each $g \in \mathcal{K}_1$, the operator $\mathcal{L}_g^\mathcal{G}$ be an isomorphism.⁴ Then there is an ε , $0 < \varepsilon < \text{dist}(\mathcal{K}_1, \mathcal{K}_2 \cup \mathcal{K}_3)/2$, such that for all $u \in \{u \in H_b^{l+2m}(G) : \text{supp } u \subset \bar{G} \cap \mathcal{O}_\varepsilon(\mathcal{K}_1)\}$ the following estimate holds:*

$$\|u\|_{H_b^{l+2m}(G)} \leq c(\|\mathbf{L}u\|_{\mathcal{H}_b^l(G, \Upsilon)} + \|u\|_{H_{b+1-l-2m}^0(G)}),$$

where $c > 0$ is independent of u .

Using the unity partition method, Leibniz' formula, Lemma 2.1 [27], and Lemma 1.2 [9], one can reduce the proof of Lemma 4.1 to the proof of the following result.

Lemma 4.2. *Let the conditions of Lemma 4.1 hold. Then for each $g \in \mathcal{K}_1$ there is an $\varepsilon_0 = \varepsilon_0(g) > 0$ such that for any $U \in \{U \in H_b^{l+2m, N}(\Omega) : \text{supp } U_j \subset \bar{\Omega}_j \cap \mathcal{V}_{\varepsilon_0}(0), j = 1, \dots, N = N(g)\}$ the following inequality holds:*

$$\|U\|_{H_b^{l+2m, N}(\Omega)} \leq c\|\mathcal{L}_g^\omega U\|_{H_b^{l, N}(\Omega)},$$

where $\mathcal{V}_{\varepsilon_0}(0) = \{x \in \mathbb{R}^n : |x| < \varepsilon_0\}$, $c > 0$ is independent of U .

Proof. Using the invertibility of $\mathcal{L}_g^\mathcal{G}$ and Lemma 3.5, for all $U \in H_b^{l+2m, N}(\Omega)$ with $\text{supp } U_j \subset \bar{\Omega}_j \cap \mathcal{V}_{\varepsilon_0}(0)$ we get

$$\begin{aligned} \|U\|_{H_b^{l+2m, N}(\Omega)} &\leq k_1\|\mathcal{L}_g^\mathcal{G}U\|_{H_b^{l, N}(\Omega)} \leq k_2(\|\mathcal{L}_g^\omega U\|_{H_b^{l, N}(\Omega)} + \\ &\quad + \varepsilon_0\|U\|_{H_b^{l+2m, N}(\Omega)} + \sum_{k=1}^N \sum_{q=2}^{R_k} \|\zeta_{qk,3}U_k - \zeta_{qk,3}\hat{U}_k\|_{H_b^{l+2m}(\Omega_k)}). \end{aligned} \quad (4.1)$$

Let us estimate the last norm in (4.1). By Theorem 4.1 [25], we have

$$\begin{aligned} \|\zeta_{qk,3}U_k - \zeta_{qk,3}\hat{U}_k\|_{H_b^{l+2m}(\Omega_k)} &\leq k_3(\|\mathcal{P}_k(\zeta_{qk,3}U_k - \zeta_{qk,3}\hat{U}_k)\|_{H_b^l(\Omega_k)} + \\ &\quad + \|\zeta_{qk,3}U_k - \zeta_{qk,3}\hat{U}_k\|_{H_{b-l-2m}^0(\Omega_k)}). \end{aligned} \quad (4.2)$$

From Lemma 3.6 and the continuity of the embedding $H_b^{l+2m}(\Omega_k) \subset H_{b-l-2m+1}^1(\Omega_k)$, it follow that

$$\|\zeta_{qk,3}U_k - \zeta_{qk,3}\hat{U}_k\|_{H_{b-l-2m}^0(\Omega_k)} \leq k_4\varepsilon_0\|U_k\|_{H_b^{l+2m}(\Omega_k)}. \quad (4.3)$$

To estimate the first norm on the right-hand side of (4.2), we apply Leibniz' formula and Lemmas 3.3 and 3.4:

$$\begin{aligned} \|\mathcal{P}_k(\zeta_{qk,3}U_k - \zeta_{qk,3}\hat{U}_k)\|_{H_b^l(\Omega_k)} &\leq k_5(\|\zeta_{qk,3}\mathcal{P}_kU_k\|_{H_b^l(\Omega_k)} + \|\zeta_{qk,3}\mathcal{P}_k\hat{U}_k\|_{H_b^l(\Omega_k)} + \\ &\quad + \sum_{|\beta| \leq 2m-1} \sum_{|\gamma| = 2m-|\beta|} \|D^\gamma \zeta_{qk,3}D^\beta U_k - D^\gamma \zeta_{qk,3}D^\beta \hat{U}_k\|_{H_b^l(\Omega_k)}) \leq k_6(\|\mathcal{P}_kU_k\|_{H_b^l(\Omega_k)} + \\ &\quad + \varepsilon_0\|U_k\|_{H_b^{l+2m}(\Omega_k)} + \sum_{|\beta| \leq 2m-1} \sum_{|\gamma| = 2m-|\beta|} \|D^\gamma \zeta_{qk,3}D^\beta U_k - D^\gamma \zeta_{qk,3}D^\beta \hat{U}_k\|_{H_b^l(\Omega_k)}). \end{aligned} \quad (4.4)$$

⁴In subsection 5 of § 1, one can find necessary and sufficient condition under which $\mathcal{L}_g^\mathcal{G}$ is an isomorphism.

Since $|D^\gamma \zeta_{kq,3}| \leq k_7 r^{-|\gamma|} |\zeta_{kq,2}|$, it follows that

$$\begin{aligned}
& \sum_{|\beta| \leq 2m-1} \sum_{|\gamma|=2m-|\beta|} \|D^\gamma \zeta_{kq,3} D^\beta U_k - D^\gamma \zeta_{kq,3} D^\beta \hat{U}_k\|_{H_b^l(\Omega_k)} \leq \\
& \leq k_8 \sum_{|\alpha| \leq l+2m-1} \|\zeta_{kq,2} D^\alpha U_k - \zeta_{kq,2} D^\alpha \hat{U}_k\|_{H_{b+|\alpha|-l-2m}^0(\Omega_k)} \leq \\
& \leq k_9 \sum_{|\alpha| \leq l+2m-1} \{ \|\zeta_{kq,2} D^\alpha U_k - \zeta_{kq,2} \widehat{D^\alpha U_k}\|_{H_{b+|\alpha|-l-2m}^0(\Omega_k)} + \\
& \quad + \|\zeta_{kq,2} \widehat{D^\alpha U_k} - \zeta_{kq,2} D^\alpha \hat{U}_k\|_{H_{b+|\alpha|-l-2m}^0(\Omega_k)} \}. \quad (4.5)
\end{aligned}$$

Using Lemma 3.6 and the continuity of the embedding $H_b^{l+2m}(\Omega_k) \subset H_{b+1+|\alpha|-l-2m}^{1+|\alpha|}(\Omega_k)$ for $|\alpha| \leq l+2m-1$, we obtain

$$\begin{aligned}
& \|\zeta_{kq,2} D^\alpha U_k - \zeta_{kq,2} \widehat{D^\alpha U_k}\|_{H_{b+|\alpha|-l-2m}^0(\Omega_k)} \leq \\
& \leq k_{10} \varepsilon_0 \|D^\alpha U_k\|_{H_{b+1+|\alpha|-l-2m}^1(\Omega_k)} \leq k_{11} \varepsilon_0 \|U_k\|_{H_b^{l+2m}(\Omega_k)}. \quad (4.6)
\end{aligned}$$

Similarly, from Lemma 3.4, it follows that

$$\|\zeta_{kq,2} \widehat{D^\alpha U_k} - \zeta_{kq,2} D^\alpha \hat{U}_k\|_{H_{b+|\alpha|-l-2m}^0(\Omega_k)} \leq k_{12} \varepsilon_0 \|U_k\|_{H_b^{l+2m}(\Omega_k)}. \quad (4.7)$$

Now the conclusion of the lemma follows from (4.1)–(4.7) with sufficiently small ε_0 . \square

2. Repeating the proof of Theorem 2.1 [11] and taking into account Lemma 5.2 [12], from Lemma 4.1 of the present work and Lemmas 2.4 and 2.5 of [11], we deduce the following result.

Theorem 4.1. *Let the conditions of Lemma 4.1 hold and $b > l+2m-1$. Then, for all $u \in H_b^{l+2m}(G)$, the following estimate holds:*

$$\|u\|_{H_b^{l+2m}(G)} \leq c(\|\mathbf{L}u\|_{\mathcal{H}_b^l(G, \Upsilon)} + \|u\|_{H_{b+1-l-2m}^0(G)}), \quad (4.8)$$

where $c > 0$ is independent of u .

By virtue of the compactness of the embedding $H_b^{l+2m}(G) \subset H_{b+1-l-2m}^0(G)$ (see Lemma 3.5 [27]), from Theorem 4.1 it follows that the operator \mathbf{L} has a finite-dimensional kernel and a closed range.

5 Construction of right regularizer

In this section, we construct a right regularizer for \mathbf{L} , which, being combined with Theorem 4.1, allows us to prove the Fredholm solvability of nonlocal boundary-value problem (1.2), (1.3).

1. To begin with, we consider the case where the supports of functions are subsets of a neighborhood of \mathcal{K}_1 . In this situation, we will use the invertibility of the operators \mathcal{L}_g^g , $g \in \mathcal{K}_1$, with linear transformations as well as some special constructions “compensating” the nonlinearity in the argument transformations. Then, in subsection 2 of this section, using the results of [11] and Lemma 5.2 [12], we will construct the right regularizer in the whole of G .

First of all, let us prove the following auxiliary result.

Lemma 5.1. *Let H , H_1 , and H_2 be Hilbert spaces, $\mathcal{A} : H \rightarrow H_1$ a linear bounded operator, $\mathcal{T}_0 : H \rightarrow H_2$ a linear compact operator. Suppose that, for some ε , $c > 0$ and all $f \in H$, the following inequality holds:*

$$\|\mathcal{A}f\|_{H_1} \leq \varepsilon\|f\|_H + c\|\mathcal{T}_0f\|_{H_2}. \quad (5.1)$$

Then there are bounded operators \mathcal{M} , $\mathcal{F} : H \rightarrow H_1$ such that

$$\mathcal{A} = \mathcal{M} + \mathcal{F},$$

where $\|\mathcal{M}\| \leq 2\varepsilon$ and the operator \mathcal{F} is finite-dimensional.

Proof. As is well known (see, e.g., [28, Chapter 5, Section 85]), any compact operator is the limit of a uniformly convergent sequence of finite-dimensional operators. Therefore, there are bounded operators \mathcal{M}_0 , $\mathcal{F}_0 : H \rightarrow H_2$ such that $\mathcal{T}_0 = \mathcal{M}_0 + \mathcal{F}_0$, $\|\mathcal{M}_0\| \leq c^{-1}\varepsilon$, and \mathcal{F}_0 is finite-dimensional. From this and (5.1), it follows that

$$\|\mathcal{A}f\|_{H_1} \leq 2\varepsilon\|f\|_H + c\|\mathcal{F}_0f\|_{H_2} \quad \text{for all } f \in H. \quad (5.2)$$

We denote by $\ker(\mathcal{F}_0)^\perp$ the orthogonal supplement in H to the kernel of \mathcal{F}_0 . Since the finite-dimensional operator \mathcal{F}_0 maps $\ker(\mathcal{F}_0)^\perp$ onto its range in a one-to-one manner, it follows that the subspace $\ker(\mathcal{F}_0)^\perp$ is of finite dimension. Let \mathcal{I} denote the unity operator in H and \mathcal{P}_0 the orthogonal projector onto $\ker(\mathcal{F}_0)^\perp$. Obviously, $\mathcal{A}\mathcal{P}_0 : H \rightarrow H_1$ is a finite-dimensional operator. Furthermore, since $\mathcal{I} - \mathcal{P}_0$ is the orthogonal projector onto $\ker(\mathcal{F}_0)$, it follows that $\mathcal{F}_0(\mathcal{I} - \mathcal{P}_0) = 0$. Therefore, substituting in (5.2) the function $(\mathcal{I} - \mathcal{P}_0)f$ for f , we get

$$\|\mathcal{A}(\mathcal{I} - \mathcal{P}_0)f\|_{H_1} \leq 2\varepsilon\|(\mathcal{I} - \mathcal{P}_0)f\|_H \leq 2\varepsilon\|f\|_H \quad \text{for all } f \in H.$$

Denoting $\mathcal{M} = \mathcal{A}(\mathcal{I} - \mathcal{P}_0)$ and $\mathcal{F} = \mathcal{A}\mathcal{P}_0$ completes the proof. \square

Now we proceed to construct the right regularizer.

Lemma 5.2. *Let the conditions of Lemma 4.1 hold. Then, for all sufficiently small ε , $0 < \varepsilon < \text{dist}(\mathcal{K}_1, \mathcal{K}_2 \cup \mathcal{K}_3)/2$, there are bounded operators \mathbf{R}_1 , \mathbf{M}_1 and a compact operator \mathbf{T}_1 acting from $\{f \in \mathcal{H}_b^l(G, \Upsilon) : \text{supp } f \subset \bar{G} \cap \mathcal{O}_\varepsilon(\mathcal{K}_1)\}$ to $H_b^{l+2m}(G)$, $\mathcal{H}_b^l(G, \Upsilon)$, and $\mathcal{H}_b^l(G, \Upsilon)$ respectively and such that*

$$\mathbf{L}\mathbf{R}_1f = f + \mathbf{M}_1f + \mathbf{T}_1f,$$

$$\|\mathbf{M}_1f\|_{\mathcal{H}_b^l(G, \Upsilon)} \leq c\varepsilon\|f\|_{\mathcal{H}_b^l(G, \Upsilon)}. \quad \text{Here } c > 0 \text{ is independent of } \varepsilon \text{ and } f.$$

Using the unity partition method, Leibniz' formula, and Lemma 2.1 [27], one can reduce the proof of Lemma 5.2 to the proof of the following result.

Lemma 5.3. *Let the conditions of Lemma 4.1 hold. Then, for each $g \in \mathcal{K}_1$ and all sufficiently small $\varepsilon_1 = \varepsilon_1(g) > 0$, there are bounded operators \mathcal{R}_g , \mathcal{M}_g and a compact operator \mathcal{T}_g acting from $\{f \in \mathcal{H}_b^{l,N}(\Omega, \Gamma) : \text{supp } f \subset \mathcal{V}_{\varepsilon_1}(0)\}$ to $H_b^{l+2m,N}(\Omega)$, $\mathcal{H}_b^{l,N}(\Omega, \Gamma)$ and $\mathcal{H}_b^{l,N}(\Omega, \Gamma)$ respectively and such that*

$$\mathcal{L}_g^\omega \mathcal{R}_g f = f + \mathcal{M}_g f + \mathcal{T}_g f, \quad (5.3)$$

$$\|\mathcal{M}_g f\|_{H_b^l(G, \Gamma)} \leq c\varepsilon_1\|f\|_{H_b^l(G, \Gamma)}. \quad \text{Here } c > 0 \text{ is independent of } \varepsilon_1 \text{ and } f.$$

Proof. 1) As before, we denote $d_1 = \min\{1, \chi_{j\sigma kqs}\}/2$, $d_2 = 2 \max\{1, \chi_{j\sigma kqs}\}$. We choose $\varepsilon_1 < d_1 \varepsilon_0/4$, where ε_0 is defined in Lemma 4.2. We introduce a function $\psi_{\varepsilon_1}(x) = \psi(x/\varepsilon_1)$, where $\psi \in C^\infty(\mathbb{R}^n)$, $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $|x| \geq 2$. It is obvious that $\psi_{\varepsilon_1} \in C^\infty(\mathbb{R}^n)$, $\psi_{\varepsilon_1}(x) = 1$ for $|x| \leq \varepsilon_1$, $\psi_{\varepsilon_1}(x) = 0$ for $|x| \geq 2\varepsilon_1$. Since $|D^\alpha \psi_{\varepsilon_1}| \leq c_\alpha r^{-|\alpha|}$, from Lemma 2.1 [27] it follows that

$$\|\psi_{\varepsilon_1} v\|_{H_b^{l+2m}(\Omega_k)} \leq c \|v\|_{H_b^{l+2m}(\Omega_k)} \quad \text{for all } v \in H_b^{l+2m}(\Omega_k), \quad (5.4)$$

where $c > 0$ is independent of ε_1 . Moreover, we assume that ψ_{ε_1} , being written in cylindrical coordinates, does not depend on φ .

Put $f_0 = \{f_j\}$, $g = \{g_{j\sigma\mu}\}$, $\{f_0, g\} = \{f_j, g_{j\sigma\mu}\}$.

By assumption, the operator $\mathcal{L}_g^{\mathcal{G}} : H_b^{l+2m,N}(\Omega) \rightarrow \mathcal{H}_b^{l,N}(\Omega, \Gamma)$ has a bounded inverse $(\mathcal{L}_g^{\mathcal{G}})^{-1} : \mathcal{H}_b^{l,N}(\Omega, \Gamma) \rightarrow H_b^{l+2m,N}(\Omega)$. Therefore, we can introduce the operators

$$\mathcal{R}_1 : H_b^{l,N}(\Omega) \rightarrow H_b^{l+2m,N}(\Omega), \quad \mathcal{R}_2 : \mathcal{H}_b^{l,N}(\Gamma) \rightarrow H_b^{l+2m,N}(\Omega)$$

given by

$$\mathcal{R}_1 f_0 = \psi_{\varepsilon_1} (\mathcal{L}_g^{\mathcal{G}})^{-1} \{f_0, 0\}, \quad \mathcal{R}_2 g = \psi_{\varepsilon_1} (\mathcal{L}_g^{\mathcal{G}})^{-1} \{0, g\},$$

where $\mathcal{H}_b^{l,N}(\Gamma) = \prod_{j,\sigma,\mu} H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})$. Thus, the supports of $\mathcal{R}_1 f_0$ and $\mathcal{R}_2 g$ are subsets of the ball of radius $2\varepsilon_1$ centered at the origin.

Let us introduce the operators

$$\begin{aligned} \mathcal{P} &: H_b^{l+2m,N}(\Omega) \rightarrow H_b^{l,N}(\Omega), \\ \mathcal{B}^{\mathcal{G}}, \mathcal{B}^{\omega} &: H_b^{l+2m,N}(\Omega) \rightarrow \mathcal{H}_b^{l,N}(\Gamma) \end{aligned}$$

given by

$$\mathcal{P}U = \{\mathcal{P}_j U_j\}, \quad \mathcal{B}^{\mathcal{G}}U = \{\mathcal{B}_{j\sigma\mu}^{\mathcal{G}} U\}, \quad \mathcal{B}^{\omega}U = \{\mathcal{B}_{j\sigma\mu}^{\omega} U\}.$$

Now we establish a relation between the operators \mathcal{P} , $\mathcal{B}^{\mathcal{G}}$, \mathcal{B}^{ω} and \mathcal{R}_1 , \mathcal{R}_2 . To this end, we will use the following well-known property of weighted spaces (see Lemma 3.5 [27]): (*) *the embedding operator from $\{v \in H_b^{l+1}(\Omega_j) : \text{supp } v \subset \mathcal{V}_d(0), d > 0\}$ into $H_b^l(\Omega_j)$ is compact.*

From Leibniz' formula, the boundedness of $\text{supp } \psi_{\varepsilon_1}$, and property (*), it follows that

$$\mathcal{P}\mathcal{R}_1 f_0 = \psi_{\varepsilon_1} f_0 + \mathcal{T}_1 f_0, \quad \mathcal{P}\mathcal{R}_2 g = \mathcal{T}_2 g, \quad (5.5)$$

where $\mathcal{T}_1 : H_b^{l,N}(\Omega) \rightarrow H_b^{l,N}(\Omega)$ and $\mathcal{T}_2 : \mathcal{H}_b^{l,N}(\Gamma) \rightarrow H_b^{l,N}(\Omega)$ are compact operators. Similarly,

$$\begin{aligned} \mathcal{B}^{\mathcal{G}}\mathcal{R}_2 g &= \psi_{\varepsilon_1} g + \\ &+ \left\{ \sum_{k,q,s} (\psi_{\varepsilon_1}(\chi_{j\sigma kqs} x) - \psi_{\varepsilon_1}(x)) (B_{j\sigma\mu kqs} [(\mathcal{L}_g^{\mathcal{G}})^{-1} \{0, g\}]_k) (\mathcal{G}_{j\sigma kqs} y, z)|_{\Gamma_{j\sigma}} \right\} + \mathcal{T}_3 g, \end{aligned} \quad (5.6)$$

where \mathcal{T}_3 is a compact operator in $\mathcal{H}_b^{l,N}(\Gamma)$; here and in what follows, we denote by $[\cdot]_k$ the k th component of an N -dimensional vector and by $\{\dots\}$ a vector with the components defined by the indices j, σ, μ .

Let us show that each term in the sum in (5.6) is a compact operator. Let $\zeta_{kq,i}$ be the functions defined by formulas (3.10). We also introduce the functions $\hat{\psi}_0, \hat{\psi}_1 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\hat{\psi}_1(x) = 1 \text{ for } 2d_1\varepsilon_1 \leq |x| \leq d_2\varepsilon_1, \quad \hat{\psi}_1(x) = 0 \text{ outside } d_1\varepsilon_1 \leq |x| \leq 2d_2\varepsilon_1,$$

$$\hat{\psi}_0(x) = 1 \text{ for } d_1\varepsilon_1 \leq |x| \leq 2d_2\varepsilon_1, \quad \hat{\psi}_0(x) = 0 \text{ outside } d_1\varepsilon_1/2 \leq |x| \leq 4d_2\varepsilon_1.$$

Then, by virtue of the boundedness of the trace operator in weighted spaces, we have

$$\begin{aligned} & \|(\psi_{\varepsilon_1}(\chi_{j\sigma kqs}x) - \psi_{\varepsilon_1}(x))(B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\}]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}}\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \leq \\ & \leq k_2\|\zeta_{kq,2}(\psi_{\varepsilon_1}(x) - \psi_{\varepsilon_1}(\chi_{j\sigma kqs}^{-1}x))B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\}]_k\|_{H_b^{l+2m-m_{j\sigma\mu}}(\Omega_k)} \leq \\ & \leq k_3\|\zeta_{kq,1}\hat{\psi}_1[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\}]_k\|_{H_b^{l+2m}(\Omega_k)}. \end{aligned} \quad (5.7)$$

Since the support of $\hat{\psi}_1$ is bounded and does not intersect with the origin and $\zeta_{kq,1}$ vanishes near the sides of the angle Ω_k , we can apply Theorem 5.1 [23, Chapter 2]. Then, using the relation $\mathcal{P}_k[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\}]_k = 0$, from (5.7) we get

$$\begin{aligned} & \|(\psi_{\varepsilon_1}(\chi_{j\sigma kqs}x) - \psi_{\varepsilon_1}(x))(B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\}]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}}\|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \leq \\ & \leq k_4\|\hat{\psi}_0[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\}]_k\|_{H_b^{l+2m-1}(\Omega_k)}. \end{aligned}$$

Since the support of $\hat{\psi}_0$ is bounded, from the last inequality and property (*) it follows that

$$\left\{ \sum_{k,q,s} (\psi_{\varepsilon_1}(\chi_{j\sigma kqs}x) - \psi_{\varepsilon_1}(x))(B_{j\sigma\mu kqs}[(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\}]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \right\}$$

is a compact operator acting in $\mathcal{H}_b^{l,N}(\Gamma)$. Combining this with (5.6) yields

$$\mathcal{B}^{\mathcal{G}}\mathcal{R}_2g = \psi_{\varepsilon_1}g + \mathcal{T}_4g, \quad (5.8)$$

where \mathcal{T}_4 is a compact operator acting in $\mathcal{H}_b^{l,N}(\Gamma)$.

Finally, from (5.8), we obtain the formula for the composition $\mathcal{B}^{\omega}\mathcal{R}_2$:

$$\begin{aligned} \mathcal{B}^{\omega}\mathcal{R}_2g &= \psi_{\varepsilon_1}g + \mathcal{T}_4g + \\ &+ \left\{ \sum_{k,q,s} \left((B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - (B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \right) \right\}. \end{aligned} \quad (5.9)$$

2) Let us introduce the operator $\mathcal{R}_g : \mathcal{H}_b^{l,N}(\Omega, \Gamma) \rightarrow H_b^{l+2m,N}(\Omega)$ given by

$$\mathcal{R}_g\{f_0, g\} = \mathcal{R}_1f_0 - \mathcal{R}'_2\mathcal{B}^{\omega}\mathcal{R}_1f_0 + \mathcal{R}_2g.$$

Here $\mathcal{R}'_2 : \mathcal{H}_b^{l,N}(\Gamma) \rightarrow H_b^{l+2m,N}(\Omega)$ is the bounded operator given by

$$\mathcal{R}'_2g = \psi_{\varepsilon_1}(d_1x/2)(\mathcal{L}_g^{\mathcal{G}})^{-1}\{0, g\}.$$

Similarly to (5.5) and (5.9), one can prove that

$$\mathcal{P}\mathcal{R}'_2g = \mathcal{T}'_2g, \quad (5.10)$$

$$\begin{aligned} \mathcal{B}^{\omega}\mathcal{R}'_2g &= \psi_{\varepsilon_1}(d_1x/2)g + \mathcal{T}'_4g + \\ &+ \left\{ \sum_{k,q,s} \left((B_{j\sigma\mu kqs}[\mathcal{R}'_2g]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - (B_{j\sigma\mu kqs}[\mathcal{R}'_2g]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \right) \right\}, \end{aligned} \quad (5.11)$$

where $\mathcal{T}'_2, \mathcal{T}'_4$ are compact operators acting in the same spaces as the operators $\mathcal{T}_2, \mathcal{T}_4$ do.

Let us show that the operator \mathcal{R}_g satisfies relation (5.3). From (5.5) and (5.10), it follows that

$$\mathcal{P}\mathcal{R}_g\{f_0, g\} = \psi_{\varepsilon_1}f_0 + \mathcal{T}_5\{f_0, g\}, \quad (5.12)$$

where $\mathcal{T}_5 : \mathcal{H}_b^{l,N}(\Omega, \Gamma) \rightarrow H_b^{l,N}(\Omega)$ is a compact operator.

Taking into account that $\psi_{\varepsilon_1}(d_1x/2)\mathcal{B}^\omega\mathcal{R}_1f_0 \equiv \mathcal{B}^\omega\mathcal{R}_1f_0$ and using (5.11), we derive

$$\begin{aligned} \mathcal{B}^\omega\mathcal{R}_g\{f_0, g\} &= \mathcal{B}^\omega\mathcal{R}_1f_0 - \mathcal{B}^\omega\mathcal{R}'_2\mathcal{B}^\omega\mathcal{R}_1f_0 + \mathcal{B}^\omega\mathcal{R}_2g = \\ &= -\mathcal{T}_4'\mathcal{B}^\omega\mathcal{R}_1f_0 - \left\{ \sum_{k,q,s} \left((B_{j\sigma\mu kqs}[\mathcal{R}'_2\mathcal{B}^\omega\mathcal{R}_1f_0]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - \right. \right. \\ &\quad \left. \left. - (B_{j\sigma\mu kqs}[\mathcal{R}'_2\mathcal{B}^\omega\mathcal{R}_1f_0]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \right) \right\} + \mathcal{B}^\omega\mathcal{R}_2g. \end{aligned}$$

From this, using (5.9), we obtain

$$\begin{aligned} \mathcal{B}^\omega\mathcal{R}_gg &= \psi_{\varepsilon_1}g + \mathcal{T}_6\{f_0, g\} + \\ &+ \left\{ \sum_{k,q,s} \left((B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - (B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \right) \right\} - \\ &- \left\{ \sum_{k,q,s} \left((B_{j\sigma\mu kqs}[\mathcal{R}'_2\mathcal{B}^\omega\mathcal{R}_1f_0]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - \right. \right. \\ &\quad \left. \left. - (B_{j\sigma\mu kqs}[\mathcal{R}'_2\mathcal{B}^\omega\mathcal{R}_1f_0]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \right) \right\}, \quad (5.13) \end{aligned}$$

where $\mathcal{T}_6 : \mathcal{H}_b^{l,N}(\Omega, \Gamma) \rightarrow \mathcal{H}_b^{l,N}(\Gamma)$ is a compact operator.

Let us consider the terms of the first sum on the right-hand side of (5.13). By Lemma 3.5, we have

$$\begin{aligned} &\| (B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - \\ &\quad - (B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \leq k_5(\varepsilon_1 \| [\mathcal{R}_2g]_k \|_{H_b^{l+2m}(\Omega_k)} + \\ &\quad + \| \zeta_{kq,3}[\mathcal{R}_2g]_k - \widehat{\zeta_{kq,3}[\mathcal{R}_2g]_k} \|_{H_b^{l+2m}(\Omega_k)}). \quad (5.14) \end{aligned}$$

From inequalities (4.2)–(4.7) for the function $U_k = [\mathcal{R}_2g]_k$, inequality (5.14), and the second relation in (5.5), we obtain

$$\begin{aligned} &\| (B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - \\ &\quad - (B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \leq \\ &\leq k_6(\varepsilon_1 \| [\mathcal{R}_2g]_k \|_{H_b^{l+2m}(\Omega_k)} + \| \mathcal{P}_k[\mathcal{R}_2g]_k \|_{H_b^l(\Omega_k)}) = \\ &= k_6(\varepsilon_1 \| \psi_{\varepsilon_1}[(\mathcal{L}_g^\mathcal{G})^{-1}\{0, g\}]_k \|_{H_b^{l+2m}(\Omega_k)} + \| [\mathcal{T}_2g]_k \|_{H_b^l(\Omega_k)}). \end{aligned}$$

This, being combined with inequality (5.4) and the boundedness of the operator $(\mathcal{L}_g^\mathcal{G})^{-1} : \mathcal{H}_b^{l,N}(\Omega, \Gamma) \rightarrow H_b^{l+2m,N}(\Omega)$, finally implies

$$\begin{aligned} &\| (B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - \\ &\quad - (B_{j\sigma\mu kqs}[\mathcal{R}_2g]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} \|_{H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})} \leq \\ &\leq k_7(\varepsilon_1 \| g \|_{\mathcal{H}_b^{l,N}(\Gamma)} + \| [\mathcal{T}_2g]_k \|_{H_b^l(\Omega_k)}). \quad (5.15) \end{aligned}$$

Therefore, by Lemma 5.1, we have

$$(B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\omega'_{j\sigma kqs}(y, z), z)|_{\Gamma_{j\sigma}} - (B_{j\sigma\mu kqs}[\mathcal{R}_2 g]_k)(\mathcal{G}_{j\sigma kqs}y, z)|_{\Gamma_{j\sigma}} = \\ = \mathcal{M}_{j\sigma\mu kqs}g + \mathcal{F}_{j\sigma\mu kqs}g$$

with the operators

$$\mathcal{M}_{j\sigma\mu kqs}, \mathcal{F}_{j\sigma\mu kqs} : \mathcal{H}_b^{l,N}(\Gamma) \rightarrow H_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_{j\sigma})$$

such that $\|\mathcal{M}_{j\sigma\mu kqs}\| \leq 2k_7\varepsilon_1$ and the operator $\mathcal{F}_{j\sigma\mu kqs}$ is finite-dimensional.

Analogously, one can prove that each term of the second sum on the right-hand side of (5.13) can be represented as the sum of an operator with small norm and a compact one. From this, (5.13), and (5.12), choosing $\text{supp}\{f_0, g\} \subset \mathcal{V}_{\varepsilon_1}(0)$, we get the conclusion of the lemma. \square

2. Now we can prove that, under certain conditions, the operator $\mathbf{L} : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \Upsilon)$ is Fredholm.

Theorem 5.1. *Let the conditions of Lemma 4.1 hold and $b > l + 2m - 1$. Then the operator $\mathbf{L} : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \Upsilon)$ is Fredholm.*

Proof. By virtue of Theorem 4.1 of the present paper and Theorems 7.1, 15.2 [29], it suffices to construct a right regularizer \mathbf{R} for \mathbf{L} .

Repeating the arguments of [11, § 3] and taking into account Lemma 5.2 [12], from Lemma 5.2 of the present paper we deduce the existence of bounded operators

$$\mathbf{R}' : \mathcal{H}_b^l(G, \Upsilon) \rightarrow H_b^{l+2m}(G), \\ \mathbf{M}, \mathbf{T} : \mathcal{H}_b^l(G, \Upsilon) \rightarrow \mathcal{H}_b^l(G, \Upsilon)$$

such that

$$\mathbf{L}\mathbf{R}' = \mathbf{I} + \mathbf{M} + \mathbf{T},$$

where $\|\mathbf{M}\| < 1$ and the operator \mathbf{T} is compact. Since $\|\mathbf{M}\| < 1$, it follows that the operator $\mathbf{I} + \mathbf{M}$ has a bounded inverse. Obviously, the operator $\mathbf{R} = \mathbf{R}'(\mathbf{I} + \mathbf{M})^{-1}$ is a right regularizer for \mathbf{L} . \square

3. Until now, we assumed that $b > l + 2m - 1$. In this subsection, using results of [9], we study the case where b is arbitrary but $n = 2$. As mentioned before, if b is arbitrary, we have to consider solutions and right-hand sides of the nonlocal problem as functions with power singularities not only near the set \mathcal{K}_1 but also near \mathcal{K}_2 and \mathcal{K}_3 . This corresponds to the consistency conditions (see § 1).

Thus, let $n = 2$. We introduce the space $\tilde{H}_b^l(G)$ as the completion of $C_0^\infty(\bar{G} \setminus \mathcal{K})$ with respect to the norm

$$\|u\|_{\tilde{H}_b^l(G)} = \left(\sum_{|\alpha| \leq l} \int_G \tilde{\rho}^{2(b-l+|\alpha|)} |D^\alpha u|^2 dy \right)^{1/2},$$

where $\tilde{\rho} = \tilde{\rho}(y) = \text{dist}(y, \mathcal{K})$ (cf. § 1). For $l \geq 1$, we denote by $\tilde{H}_b^{l-1/2}(\Upsilon)$ the space of traces on a smooth curve $\Upsilon \subset \bar{G}$ with the norm

$$\|\psi\|_{\tilde{H}_b^{l-1/2}(\Upsilon)} = \inf \|u\|_{\tilde{H}_b^l(G)} \quad (u \in \tilde{H}_b^l(G) : u|_\Upsilon = \psi).$$

We assume that the following condition holds.

Condition 5.1. *If $g \in \mathcal{K}_3 \cap \omega_{is}(\Upsilon_i) \neq \emptyset$, then $\omega_{is}^{-1}(g) \in \mathcal{K}$.*

The fulfillment of Condition 5.1 guarantees that the set of points in which the consistency condition must be imposed is finite. If Condition 5.1 fails, then the consecutive shifts of the set \mathcal{K}_1 (under the transformations ω_{is} and ω_{is}^{-1}) may form an infinite set, which should be used instead of \mathcal{K} in the definition of weighted spaces.

In this subsection, we consider the following bounded operator corresponding to problem⁵ (1.2), (1.3):

$$\mathbf{L} = \{\mathbf{P}(y, D), \mathbf{B}_{i\mu}(y, D)\} : \tilde{H}_b^{l+2m}(G) \rightarrow \tilde{H}_b^l(G) \times \prod_{i=1}^{N_0} \prod_{\mu=1}^m \tilde{H}_b^{l+2m-m_{i\mu}-1/2}(\Upsilon_i), \quad b \in \mathbb{R}.$$

Since solutions and right-hand sides of the nonlocal problem may now have power singularities near the points of \mathcal{K}_2 and \mathcal{K}_3 , we have to consider the model problems corresponding to these points in weighted spaces but not in the Sobolev spaces.

We fix a point $g \in \mathcal{K}_2 \cup \mathcal{K}_3$. Let $y \mapsto y'(g)$ be a non-degenerate infinitely differentiable argument transformation mapping some neighborhood $\mathcal{V}(g)$ of the point g onto a neighborhood $\mathcal{V}_g(0)$ of the origin, so that the point g maps to the origin. We denote by $\mathcal{P}(D_y)$, $\mathcal{B}_{i\mu 0}(D_y)$ the principal homogeneous parts of the operators $\mathbf{P}(g, D)$, $\mathbf{B}_{i\mu 0}(g, D)$ written in new coordinates $y' = y'(g)$ (with after-denoting y' by y). Now we write the operators $\mathcal{P}(D_y)$, $\mathcal{B}_{i\mu 0}(D_y)$ in polar coordinates: $\mathcal{P}(D_y) = r^{-2m} \tilde{\mathcal{P}}(\varphi, D_\varphi, rD_r)$, $\mathcal{B}_{i\mu 0}(D_y) = r^{-m_{i\mu}} \tilde{\mathcal{B}}_{i\mu 0}(\varphi, D_\varphi, rD_r)$.

If $g \in \mathcal{K}_2$, then $g \in \Upsilon_i$ for some $i = i(g)$. By virtue of the smoothness of Υ_i , in a sufficiently small neighborhood $\mathcal{V}(g)$ of g there is a non-degenerate infinitely smooth argument transformation $y \mapsto y' = y'(g)$ mapping $\mathcal{V}(g) \cap G$ onto the intersection of the half-plane $\mathbb{R}_+^2 = \{y : |\varphi| < \pi/2\}$ with a neighborhood of $\mathcal{V}_g(0)$. Let us introduce the bounded operator

$$\mathcal{L}_g : H_b^{l+2m}(K_{\pi/2}) \rightarrow H_b^l(K_{\pi/2}) \times \prod_{j=1}^2 \prod_{\mu=1}^m H_b^{l+2m-m_{i\mu}-1/2}(\gamma_j)$$

given by

$$\mathcal{L}_g U = \{\mathcal{P}(D_y)U, \mathcal{B}_{i\mu 0}(D_y)U|_{\gamma_j}\},$$

where $K_{\pi/2} = \{y : |\varphi| < \pi/2\}$, $\gamma_j = \{y : \varphi = (-1)^j \pi/2\}$, $j = 1, 2$. We also introduce the bounded operator

$$\tilde{\mathcal{L}}_g(\lambda) : W_2^{l+2m}(-\pi/2, \pi/2) \rightarrow \mathcal{W}_2^l[-\pi/2, \pi/2] = W_2^l(-\pi/2, \pi/2) \times \mathbb{C}^{2m}$$

given by

$$\tilde{\mathcal{L}}_g(\lambda) \tilde{U} = \{\tilde{\mathcal{P}}(\varphi, D_\varphi, \lambda) \tilde{U}(\varphi), \tilde{\mathcal{B}}_{i\mu 0}(\varphi, D_\varphi, \lambda) \tilde{U}(\varphi)|_{\varphi=(-1)^j \pi/2}\}, \quad j = 1, 2.$$

If $g \in \mathcal{K}_3$, we introduce the bounded operator

$$\mathcal{L}_g = \mathcal{P}(D_y) : H_b^{l+2m}(\mathbb{R}^2) \rightarrow H_b^l(\mathbb{R}^2).$$

Let us also introduce the bounded operator

$$\tilde{\mathcal{L}}_g(\lambda) = \tilde{\mathcal{P}}(\varphi, D_\varphi, \lambda) : W_{2,2\pi}^{l+2m}(0, 2\pi) \rightarrow W_{2,2\pi}^l(0, 2\pi),$$

⁵Notice that equation (1.2) is now considered in $G \setminus \mathcal{K}_3$ but not in the whole of G .

where $W_{2,2\pi}^l(0, 2\pi)$ is the closure of the set of infinitely differentiable 2π -periodic functions in $W_2^l(0, 2\pi)$.

From [27, § 1] and [9, § 1], it follows that for each $g \in \mathcal{K}_2 \cup \mathcal{K}_3$ there is a finite-meromorphic operator-valued function $\tilde{\mathcal{L}}_g^{-1}(\lambda)$ such that (I) its poles, maybe with the exception of finitely many of them, belong to a double angle of opening $< \pi$, containing the imaginary axis, and (II) for a λ which is not a pole of $\tilde{\mathcal{L}}_g^{-1}(\lambda)$, the operator $\tilde{\mathcal{L}}_g^{-1}(\lambda)$ is the bounded inverse for $\tilde{\mathcal{L}}_g(\lambda)$.

From Theorem 1.1 [27] and results of [9, § 1], it follows that the operator \mathcal{L}_g is an isomorphism if and only if the line $\text{Im } \lambda = b + 1 - l - 2m$ contains no poles of $\tilde{\mathcal{L}}_g^{-1}(\lambda)$.

Theorem 5.2. *Let Conditions 1.1–1.4 and 5.1 hold. Suppose that $b \in \mathbb{R}$ is such that for all $g \in \mathcal{K}_1$ the operator $\mathcal{L}_g^{\mathcal{G}}$ is an isomorphism and for all $g \in \mathcal{K}_2 \cup \mathcal{K}_3$ the operator \mathcal{L}_g is an isomorphism.*

Then the operator $\mathbf{L} : \tilde{H}_b^{l+2m}(G) \rightarrow \tilde{\mathcal{H}}_b^l(G, \Upsilon)$ is Fredholm.

Proof. Notice that Lemmas 4.1 and 5.2 are true for any $b \in \mathbb{R}$ for which the operators $\mathcal{L}_g^{\mathcal{G}}$, $g \in \mathcal{K}_1$, are isomorphisms. Therefore, using Lemmas 4.1 and 5.2, analogously to the proof of Theorem 3.4 [9], we can obtain an a priori estimate (4.8) (in the spaces $\tilde{H}_b^l(\cdot)$) and construct a right regularizer. \square

6 Index stability for nonlocal elliptic problems

In this section, we study an influence of the transformations ω_{is} upon the index of nonlocal elliptic problems. We show that the index of the problem is determined by the linear part of the transformations ω_{is} in a neighborhood of \mathcal{K}_1 . Notice that, in the case where the support $\bigcup_{i,s} \omega_{is}(\tilde{\Upsilon}_i)$ of nonlocal terms does not intersect with the set \mathcal{K}_1 consisting of the points of conjugation of nonlocal conditions, the index stability for the corresponding problem was proved in [15].

1. Parallel to problem (1.2), (1.3), we consider the following problem:

$$\mathbf{P}(x, D)u = f_0(x) \quad (x \in G), \quad (6.1)$$

$$\begin{aligned} \hat{B}_{i\mu}(x, D)u &\equiv \sum_{s=0}^{\hat{S}_i} (\hat{B}_{i\mu s}(x, D)u)(\hat{\omega}_{is}(x))|_{\Upsilon_i} = g_{i\mu}(x) \\ (x \in \Upsilon_i; \ i = 1, \dots, N_0; \ \mu = 1, \dots, m). \end{aligned} \quad (6.2)$$

Here $\mathbf{P}(x, D)$, $\hat{B}_{i\mu 0}(x, D) = B_{i\mu 0}(x, D)$ are the same⁶ differential operators as those in § 1, $\hat{B}_{i\mu s}(x, D)$ ($s = 1, \dots, \hat{S}_i$) are some differential operators of orders $m_{i\mu}$ with complex-valued C^∞ -coefficients; $\hat{\omega}_{is}$ ($i = 1, \dots, N_0; s = 1, \dots, \hat{S}_i$) are infinitely differentiable non-degenerate transformations mapping some neighborhood \mathcal{O}_i of the manifold Υ_i onto $\hat{\omega}_{is}(\mathcal{O}_i)$ so that $\hat{\omega}_{is}(\Upsilon_i) \subset G$; $\omega_{i0}(x) \equiv x$. We assume that the set

$$\hat{\mathcal{K}} = \left\{ \bigcup_i (\tilde{\Upsilon}_i \setminus \Upsilon_i) \right\} \cup \left\{ \bigcup_{i,s} \hat{\omega}_{is}(\tilde{\Upsilon}_i \setminus \Upsilon_i) \right\} \cup \left\{ \bigcup_{j,p} \bigcup_{i,s} \hat{\omega}_{jp}(\hat{\omega}_{is}(\tilde{\Upsilon}_i \setminus \Upsilon_i) \cap \Upsilon_j) \right\}$$

⁶It suffices that only the principal homogeneous parts of the operators $\mathbf{P}(x, D)$ and $\hat{B}_{i\mu 0}(x, D)$ from this section and those from § 1 coincide. But, for simplicity, we assume that junior terms of the corresponding operators also coincide.

can be represented in the form $\hat{\mathcal{K}} = \bigcup_{j=1}^3 \bigcup_{p=1}^{\hat{N}_j} \hat{\mathcal{K}}_{jp}$, where

$$\hat{\mathcal{K}}_1 = \bigcup_{p=1}^{\hat{N}_1} \hat{\mathcal{K}}_{1p} = \partial G \setminus \bigcup_{i=1}^{N_0} \Upsilon_i, \quad \hat{\mathcal{K}}_2 = \bigcup_{p=1}^{\hat{N}_2} \hat{\mathcal{K}}_{2p} \subset \bigcup_{i=1}^{N_0} \Upsilon_i, \quad \hat{\mathcal{K}}_3 = \bigcup_{p=1}^{\hat{N}_3} \hat{\mathcal{K}}_{3p} \subset G$$

(cf. (1.1)). Here $\hat{\mathcal{K}}_{jp}$ are disjoint $(n-2)$ -dimensional C^∞ -manifolds without a boundary (points if $n=2$); moreover, $\hat{N}_1 = N_1$, $\hat{\mathcal{K}}_{1p} = \mathcal{K}_{1p}$, $p=1, \dots, N_1$.

Let the transformations $\hat{\omega}_{is}$ satisfy Conditions 1.3 and 1.4. Furthermore, we assume that the operators $\hat{B}_{i\mu s}(x, D)$ and the transformations $\hat{\omega}_{is}$ ($s=1, \dots, \hat{S}_i$) are such that for each $g \in \hat{\mathcal{K}}_1 = \mathcal{K}_1$ the operator $\mathcal{L}_g^{\hat{\omega}}$ (which is defined similarly to the operator \mathcal{L}_g^ω from § 1) equals the operator \mathcal{L}_g^g defined in § 1.

Thus, $\hat{\omega}_{is}$ is a linear part of ω_{is} in a neighborhood of \mathcal{K}_1 .

We introduce the bounded operator corresponding to nonlocal problem (6.1), (6.2):

$$\hat{\mathbf{L}} = \{\mathbf{P}(x, D), \hat{\mathbf{B}}_{i\mu}(x, D)\} : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \Upsilon).$$

Theorem 6.1. *Let the conditions of Lemma 4.1 hold and $b > l + 2m - 1$. Then the operators $\mathbf{L}, \hat{\mathbf{L}} : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \Upsilon)$ are Fredholm and $\text{ind } \mathbf{L} = \text{ind } \hat{\mathbf{L}}$.*

Proof. We consider the operator $\mathbf{L}_t : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \Upsilon)$ given by

$$\mathbf{L}_t u = \{\mathbf{P}(x, D)u, \mathbf{B}_{i\mu}(x, D) + t(\hat{\mathbf{B}}_{i\mu}(x, D) - \mathbf{B}_{i\mu}(x, D))\}.$$

Obviously, $\mathbf{L}_0 = \mathbf{L}$, $\mathbf{L}_1 = \hat{\mathbf{L}}$.

In a neighborhood of \mathcal{K}_1 , the transformations ω_{is} and $\hat{\omega}_{is}$ coincide up to infinitesimals; therefore, by Theorem 5.1, the operators \mathbf{L}_t are Fredholm for all t . Furthermore, for all t_0 and t , we have

$$\|\mathbf{L}_t u - \mathbf{L}_{t_0} u\|_{\mathcal{H}_b^l(G, \Upsilon)} \leq k_{t_0} |t - t_0| \cdot \|u\|_{H_b^{l+2m}(G)},$$

where $k_{t_0} > 0$ is independent of $t \in [0, 1]$. Hence, by Theorem 16.2 [29], we have $\text{ind } \mathbf{L}_t = \text{ind } \mathbf{L}_{t_0}$ for all t from some small neighborhood of t_0 . These neighborhoods cover the segment $[0, 1]$. Choosing a finite subcovering, we get $\text{ind } \mathbf{L} = \text{ind } \mathbf{L}_0 = \text{ind } \mathbf{L}_1 = \text{ind } \hat{\mathbf{L}}$. \square

Analogously to the above, using Theorem 5.2 instead of Theorem 5.1, one can prove the index stability for nonlocal problem (1.2), (1.3) in the case where $n=2$, $b \in \mathbb{R}$.

Let us suppose that $\hat{N}_j = N_j$, $\hat{\mathcal{K}}_{jp} = \mathcal{K}_{jp}$, $j=1, 2, 3$, $p=1, \dots, N_j$.

Theorem 6.2. *Let the conditions of Theorem 5.2 hold. Then the operators $\mathbf{L}, \hat{\mathbf{L}} : \tilde{H}_b^{l+2m}(G) \rightarrow \tilde{\mathcal{H}}_b^l(G, \Upsilon)$ are Fredholm and $\text{ind } \mathbf{L} = \text{ind } \hat{\mathbf{L}}$.*

2. In this subsection, we present another proof of Theorem 6.2, based upon ideas of [15]. (Notice that, using Lemma 5.2 [12], one can similarly prove Theorem 6.1.) The proof given below is more complicated; however it makes clear the phenomenon—*why index of the operator is completely determined by the linear part of the transformations ω_{is} in a neighborhood of \mathcal{K}_1* . We show that if the operators \mathbf{L} and $\hat{\mathbf{L}}$ are both Fredholm, then the restriction of their difference to the kernel $\ker(\mathbf{P}) \subset \tilde{H}_b^{l+2m}(G)$ of the operator $\mathbf{P} = \mathbf{P}(y, D)$ (we remind that $x=y$ if $n=2$) can be “reduced” to the sum of an operator with an arbitrary small norm and an operator the

square of which is compact. The first operator appears at the expense of the nonlinear part of the transformations ω_{is} near \mathcal{K}_1 while the second one appears at the expense of transformations originating the sets \mathcal{K}_2 and \mathcal{K}_3 (see § 1). Notice that this “reduction” does not contradict the example of § 2 since the “reduction” procedure contains projecting to the subspace $\ker(\mathbf{P})$ of infinite codimension. By the same reason, the considerations below do not prove that the operator $\hat{\mathbf{L}}$ is Fredholm whenever \mathbf{L} is Fredholm (or vice versa). The only thing they imply is that $\text{ind } \mathbf{L} = \text{ind } \hat{\mathbf{L}}$ whenever we are a priori aware of \mathbf{L} and $\hat{\mathbf{L}}$ being both Fredholm.

Thus, let us proceed to the alternative proof of Theorem 6.2.

1) We introduce the operators

$$\mathbf{B}, \hat{\mathbf{B}} : \tilde{H}_b^{l+2m}(G) \rightarrow \tilde{\mathcal{H}}_b^l(\partial G) = \prod_{i=1}^{N_0} \prod_{\mu=1}^m \tilde{H}_b^{l+2m-m_{i\mu}-1/2}(\Upsilon_i)$$

given by $\mathbf{B} = \{\mathbf{B}_{i\mu}(y, D)\}$, $\hat{\mathbf{B}} = \{\hat{\mathbf{B}}_{i\mu}(y, D)\}$. We denote by $\mathbf{C}, \hat{\mathbf{C}}$ the restrictions of the operators $\mathbf{B}, \hat{\mathbf{B}}$ to the subspace $\ker(\mathbf{P}) \subset \tilde{H}_b^{l+2m}(G)$. By Theorem 5.1, the operators $\mathbf{L}, \hat{\mathbf{L}}$ are Fredholm. Therefore, by virtue of Lemma 1.1 [15], the operators $\mathbf{C}, \hat{\mathbf{C}}$ are also Fredholm. Now, to prove Theorem 6.2, it suffices to show that $\text{ind } \mathbf{C} = \text{ind } \hat{\mathbf{C}}$.

2) We denote by $\mathbf{C}^1, \hat{\mathbf{C}}^1$ the restrictions of $\mathbf{C}, \hat{\mathbf{C}}$ to the subspace $\ker(\mathbf{C})^\perp \subset \ker(\mathbf{P})$. It is obvious that $\mathbf{C}^1 = \mathbf{C}\mathbf{I}_0$, $\hat{\mathbf{C}}^1 = \hat{\mathbf{C}}\mathbf{I}_0$, where $\mathbf{I}_0 : \ker(\mathbf{C})^\perp \rightarrow \ker(\mathbf{P})$ is the operator of embedding of $\ker(\mathbf{C})^\perp$ into $\ker(\mathbf{P})$. Clearly, we have $\dim \ker(\mathbf{I}_0) = 0$, $\text{codim } \mathcal{R}(\mathbf{I}_0) = \dim \ker(\mathbf{C}) = m_0 < \infty$. Therefore, from Theorem 12.2 [29], it follows that

$$\begin{aligned} \text{ind } \mathbf{C}^1 &= \text{ind } \mathbf{C} + \text{ind } \mathbf{I}_0 = \text{ind } \mathbf{C} - m_0, \\ \text{ind } \hat{\mathbf{C}}^1 &= \text{ind } \hat{\mathbf{C}} + \text{ind } \mathbf{I}_0 = \text{ind } \hat{\mathbf{C}} - m_0. \end{aligned}$$

Thus, it suffices to prove that $\text{ind } \mathbf{C}^1 = \text{ind } \hat{\mathbf{C}}^1$.

3) We denote by \mathbf{P}_\perp the operator that orthogonally projects $\tilde{\mathcal{H}}_b^l(\partial G)$ onto $\mathcal{R}(\mathbf{C}^1)^\perp$. Since $\text{codim } \mathcal{R}(\mathbf{C}^1) < \infty$, it follows that the operator \mathbf{P}_\perp is finite-dimensional. Therefore, we have

$$\text{ind } \hat{\mathbf{C}}^1 = \text{ind } (\mathbf{C}^1 + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)).$$

Hence, it suffices to prove that

$$\text{ind } \mathbf{C}^1 = \text{ind } (\mathbf{C}^1 + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)).$$

Since $\mathbf{C}^1 u, \mathbf{C}^1 u + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)u \in \mathcal{R}(\mathbf{C}^1)$ for $u \in \ker(\mathbf{C})^\perp$, we may regard $\mathbf{C}^1, \mathbf{C}^1 + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)$ as the operators acting from $\ker(\mathbf{C})^\perp$ into $\mathcal{R}(\mathbf{C}^1)$. In this case, the indices of these operators increase the same number $m_1 = \text{codim } \mathcal{R}(\mathbf{C}^1)$.

Evidently, the operator $\mathbf{C}^1 : \ker(\mathbf{C})^\perp \rightarrow \mathcal{R}(\mathbf{C}^1)$ has the bound inverse $\mathbf{R}_1 = (\mathbf{C}^1)^{-1} : \mathcal{R}(\mathbf{C}^1) \rightarrow \ker(\mathbf{C})^\perp$ and $\text{ind } \mathbf{C}^1 = 0$. By Theorem 12.2 [29], we have

$$\text{ind } (\mathbf{C}^1 + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)) = \text{ind } (\mathbf{I} + \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)).$$

It remains to show that $\text{ind } (\mathbf{I} + \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)) = 0$.

4) Let us introduce a function $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^2)$ such that $\psi_\varepsilon(y) = 1$ for $y \in \mathcal{O}_{\varepsilon/2}(\mathcal{K})$, $\psi_\varepsilon(y) = 0$ for $y \notin \mathcal{O}_\varepsilon(\mathcal{K})$, and

$$|D^\alpha \psi_\varepsilon(y)| \leq k_\alpha (\tilde{\rho}(y))^{-|\alpha|} \quad (y \in \mathcal{O}_\varepsilon(\mathcal{K})), \quad (6.3)$$

where $k_\alpha > 0$ is independent of ε .

We consider the operators $\mathbf{A}_1, \mathbf{A}_2 : \ker(\mathbf{C})^\perp \rightarrow \ker(\mathbf{C})^\perp$ given by

$$\begin{aligned}\mathbf{A}_1 u &= \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{B}} - \mathbf{B})\psi_\varepsilon u, \\ \mathbf{A}_2 u &= \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{B}} - \mathbf{B})(1 - \psi_\varepsilon)u.\end{aligned}$$

It is clear that $\mathbf{I} + \mathbf{A}_1 + \mathbf{A}_2 = \mathbf{I} + \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)$. Since the support of $(1 - \psi_\varepsilon)u$ does not intersect with the origin, it follows from the proof of Theorem 3.1 [15] that the operator $(\mathbf{A}_2)^2$ is compact.

Let us study the operator \mathbf{A}_1 . Since the operator $\mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)$ is bounded, it follows that

$$\|\mathbf{A}_1 u\|_{\tilde{H}_b^{l+2m}(G)} \leq c \|(\hat{\mathbf{B}} - \mathbf{B})\psi_\varepsilon u\|_{\tilde{H}_b^l(\partial G)}.$$

From this, using the unity partition method and estimates (4.2)–(4.7), followed by (6.3), we obtain

$$\begin{aligned}\|\mathbf{A}_1 u\|_{\tilde{H}_b^{l+2m}(G)} &\leq c_1(\varepsilon \|\psi_\varepsilon u\|_{\tilde{H}_b^{l+2m}(G)} + \|\mathbf{P}\psi_\varepsilon u\|_{\tilde{H}_b^l(G)}) + k_1(\varepsilon) \|u\|_{\tilde{H}_b^{l+2m-1}(G)} \leq \\ &\leq c_2(\varepsilon \|u\|_{\tilde{H}_b^{l+2m}(G)} + \|\mathbf{P}\psi_\varepsilon u\|_{\tilde{H}_b^l(G)}) + k_1(\varepsilon) \|u\|_{\tilde{H}_b^{l+2m-1}(G)}.\end{aligned}\quad (6.4)$$

Since $u \in \ker(\mathbf{P})$, from (6.4) and Leibniz' formula, we get

$$\|\mathbf{A}_1 u\|_{\tilde{H}_b^{l+2m}(G)} \leq c_2 \varepsilon \|u\|_{\tilde{H}_b^{l+2m}(G)} + k_2(\varepsilon) \|u\|_{\tilde{H}_b^{l+2m-1}(G)}, \quad (6.5)$$

where c_2 is independent of ε . From (6.5), the compactness of the embedding $\tilde{H}_b^{l+2m}(G) \subset \tilde{H}_b^{l+2m-1}(G)$, and Lemma 5.1, it follows that $\mathbf{A}_1 = \mathbf{M}_1 + \mathbf{F}_1$, where $\|\mathbf{M}_1\| \leq 2c_2\varepsilon$ and the operator \mathbf{F}_1 is finite-dimensional.

Thus, we have $\mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1) = \mathbf{M}_1 + \mathbf{F}_1 + \mathbf{A}_2$. Therefore, choosing sufficiently small ε , we obtain from Theorems 15.4 and 16.2 [29] that $\text{ind}(\mathbf{I} + \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}^1 - \mathbf{C}^1)) = 0$.

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References

- [1] Sommerfeld A. Ein Beitrag zur hydrodinamischen Erklärung der turbulenten Flüssigkeitsbewegungen, *Proc. Intern. Congr. Math. (Rome, 1908)*. 1909. V. 3. Reale Accad. Lincei. Roma. P. 116–124.
- [2] Tamarkin J.D. *Some General Problems of the Theory of Ordinary Linear Differential Equations and Expansion of an Arbitrary Function in Series of Fundamental Functions*, Petrograd, 1917. Abridged English transl. in *Math. Z.* 1928. V. 27. P. 1–54.
- [3] Picone M. Equazione integrale traducente il più generale problema lineare per le equazioni differenziali lineari ordinarie di qualsivoglia ordine, *Accademia nazionale dei Lincei. Atti dei convegni*. 1932. V. 15. P. 942–948.
- [4] Carleman T. Sur la théorie des equations integrales et ses applications, *Verhandlungen des Internat. Math. Kongr. Zürich*. 1932. V. 1. P. 132–151.

- [5] Bitsadze A.V., Samarskii A.A. On some simple generalizations of linear elliptic boundary value problems, *Dokl. Akad. Nauk SSSR*. 1969. V. 185. P. 739–740; English transl. in *Soviet Math. Dokl.* 1969. V. 10.
- [6] Bitsadze A.V. On some class of conditionally solvable nonlocal boundary value problems for harmonic functions, *Dokl. Akad. Nauk SSSR*. 1985. V. 280, No. 3. P. 521–524. English transl. in *Soviet Math. Dokl.* 1985. V. 31.
- [7] Kishkis K.Yu. The index of a Bitsadze–Samarskii Problem for harmonic functions, *Differentsial'nye Uravneniya*. 1988. V. 24. No. 1. P. 105–110. English transl. in *Differential Equations*. 1988. V. 24.
- [8] Gushchin A.K., Mikhailov V.P. On solvability of nonlocal problems for elliptic equations of second order, *Mat. sb.* 1994. V. 185. P. 121–160; English transl. in *Math. Sb.* 1994. V. 185.
- [9] Skubachevskii A.L. Elliptic problems with nonlocal conditions near the boundary, *Mat. Sb.* 1986. V. 129 (171). P. 279–302. English transl. in *Math. USSR-Sb.* 1987. V. 57.
- [10] Skubachevskii A.L. Model nonlocal problems for elliptic equations in dihedral angles, *Differentsial'nye Uravneniya*. 1990. V. 26, No. 1. P. 120–131. English transl. in *Differential Equations*. 1990. V. 26.
- [11] Skubachevskii A.L. Truncation-function method in the theory of nonlocal problems, *Differentsial'nye Uravneniya*. 1991. V. 27, No. 1. P. 128–139. English transl. in *Differential Equations*. 1991. V. 27.
- [12] Kovaleva O.A., Skubachevskii A.L. Solvability of nonlocal elliptic problems in weighted spaces, *Mat. Zametki*. 2000. V. 67. P. 882–898. English transl. in *Math. Notes*. 2000. V. 67.
- [13] Gurevich P.L. Nonlocal problems for elliptic equations in dihedral angles and the Green formula. *Mitteilungen aus dem Mathem. Seminar Giessen*. Math. Inst. Univ. Giessen, Germany, V. 247. 2001. P. 1–74.
- [14] Gurevich P.L. Asymptotics of solutions for nonlocal elliptic problems in plane angles, *Trudy seminara imeni I.G. Petrovskogo*. 2003. V. 23. English transl. in *J. Math. Sci., New York*. 2003.
- [15] Skubachevskii A.L. On the stability of index of nonlocal elliptic problems, *Journal of Mathematical Analysis and Applications*. 1991. V. 160. No 2. P. 323–341.
- [16] Samarskii A.A. On some problems of theory of differential equations, *Differentsial'nye Uravneniya*. 1980. V. 16. No. 11. P. 1925–1935. English transl. in *Differential Equations*. 1980. V. 16.
- [17] Feller W. The parabolic differential equations and the associated semi-groups of transformations, *Ann. of Math.* 1952. V. 55. P. 468–519.
- [18] Feller W. Diffusion processes in one dimension, *Trans. Amer. Math. Soc.* 1954. V. 77. P. 1–30.
- [19] Taira K. On the existence of Feller semigroups with boundary conditions, *Mem. Amer. Math. Soc.* 1992. V. 99. P. 1–65.

- [20] Ventsel' A.D. On boundary conditions for multidimensional diffusion processes, *Teoriya Veroyatn. i ee Primen.* 1959. V. 4. P. 172–185; English transl. in *Theory Prob. and its Appl.* 1959. V. 4.
- [21] Skubachevskii A.L. *Elliptic Functional Differential Equations and Applications*. Basel–Boston–Berlin, Birkhäuser. 1997.
- [22] Onanov G.G., Skubachevskii A.L. Differential equations with displaced arguments in stationary problems in the mechanics of a deformed body, *Prikladnaya Mekhanika*. 1979. V. 15. P. 39–47; English transl. in *Soviet Applied Mech.* 1979. V. 15.
- [23] Lions J.L., Magenes, E. *Problèmes aux Limites Non-Homogènes et Applications*. V. I, Dunod, Paris. 1968. English translation: in Springer–Verlag. 1972.
- [24] Stein E.M. *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.
- [25] Maz'ya V.G., Plamenevskii B.A., L_p -estimates of solutions of elliptic boundary value problems in domains with edges, *Trudy Moskov. Mat. Obshch.* 1978. V. 37. P. 49–93. English transl. in *Trans. Moscow Math. Soc.* 1980. V. 37.
- [26] Slobodetskii L.N. Generalized Sobolev spaces and their application to boundary problems for partial differential equations, *Leningrad. Gos. Ped. Inst. Uchen. Zap.* 1958. V. 197. P. 54–112. English transl. in *Amer. Math. Soc. Transl.* (2). 1966. V. 57.
- [27] Kondrat'ev V.A. Boundary value problems for elliptic equations in domains with conical or angular points, *Trudy Moskov. Mat. Obshch.* 1967. V. 16. P. 209–292. English transl. in *Trans. Moscow Math. Soc.* 1967. V. 16.
- [28] Riesz F, Sz.-Nagy. *Leçons d'Analyse Fonctionnelle*. Deuxième édition. Budapest. 1953.
- [29] Krein S.G. *Linear Equations in Banach Spaces*, Nauka, Moscow, 1971 (Russian). English translation: Birkhäuser, Boston, 1982.